PRINCIPLES

OF

ARITHMETIC.

O'SULLIVAN.

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THE PRINCIPLES OF ARITHMETIC:

A Comprehensive Text-Book

FOR THE USE OF

TEACHERS AND ADVANCED PUPILS.

BY

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PREFACE.

THIS book, which is detailed as well as comprehensive, and which is the result of a long and varied experience, will, I hope, enable persons of ordinary capacity and intelligence to acquire for themselves, by diligent and patient study, a sound knowledge of the *rationale* of Arithmetic. The want of such a treatise has, in numerous instances, been complained of by the Irish National Teachers, with whom—as Inspector of National Schools, as Head Master of the Central Model Schools, and as Professor—I have, for many years, been officially connected.

In the following pages, Arithmetic is reduced to a series of distinct "Principles," which—for facility of reference—are printed in large type, and numbered. The student is asked to take nothing for granted. Every Principle is logically established, and is, moreover, illustrated by a sufficient number of carefully selected examples.

In the arrangement and treatment of the Principles, I have exercised my own judgment. Many

of the demonstrations are original; and some of them, in which the employment of figures is supplemented by that of general symbols, may be acceptable to the Algebraist, as well as to the Arithmetician.

Although written with especial reference to the wants of the National Teachers, the PRINCIPLES OF ARITHMETIC will be found to embrace everything upon which, under the head of Arithmetic, candidates for Civil Service appointments are examined.

D. O'SULLIVAN.

DUBLIN,
August, 1872.

CONTENTS.

ARITHMETIC.

DEFINITIONS.

1. Any *one* thing, regarded as to its singleness, is termed a UNIT.

2. Two or more units of the same kind, considered collectively, are called a NUMBER.

We say "a *number* of birds," "a *number* of trees," "a *number* of stars," &c.: the unit (whose repetition constitutes the number) being—in the first case, a bird; in the second, a tree; in the third, a star; &c.

Almost as soon as it can observe anything, a child is able to distinguish between "one" object and a "number" of such objects—between, for example, one apple and a number of apples, between one marble and a number of marbles, &c. Further observation enables the child in a very short time to distinguish between a larger and a smaller number of objects, all of the same kind—between, say, five apples and three apples, between ten marbles and seven marbles, &c. Comparing a number of objects of one kind with the *same* number of objects of another kind (a number of apples, for instance, with the same number of marbles), the child soon begins to see that the mere *number* is quite distinct from the *nature* of the objects—in other words, that the perception of *what* the objects are is quite distinct from that of *how many* there are; and so the mind gradually acquires the power of contemplating numbers without reference to any objects in particular—or, as we say, of contemplating numbers in the *abstract*. Thus, a boy who has three apples in one pocket, and three marbles in another, cannot fail to see that, although an apple and a marble are two very different things, the *number* of apples is the same as the *number* of marbles; and the contemplation of this sameness gives the boy the *abstract* notion, as it is called, of the number "three."

3. A number is said to be CONCRETE or ABSTRACT according as it is considered with or without reference to particular things.

"Five *books*," "eight *shillings*," "ten *days*," &c. are examples of *concrete* numbers—mention being, in each case, made of *what* the units are, as well as of *how many* there are; but "five," "eight," "ten," &c.—considered without reference to any units in particular—are *abstract* numbers.

Although the word "number" conveys the idea of plurality, it is not unusual to speak of "the number *one*"—just as we say "the number three," "the number five," &c.: "one" being popularly regarded as the smallest number. On the other hand, the term "unit" is sometimes applied (but only in a secondary sense) to a *collection* of individuals. Thus, when mention is made of "five regiments" of soldiers, the mind is disposed to regard five as the "number," and a *regiment* as the "unit;" when we speak of "eight mease" of herrings, eight may be regarded as the "number," and a *mease* (five hundred) as the "unit;" and when we say "ten dozen" of wine, ten may be taken as the "number," and a *dozen* (bottles) as the "unit."

4. ARITHMETIC is the Science of Numbers.

NOTATION AND NUMERATION.

DECIMAL OR ARABIC SYSTEM.*

5. Every number can be expressed by means of one or more of the following ten figures, the first nine of which are called *digits*:—

1	2	3	4	5	6	7	8	9	0
one	two	three	four	five	six	seven	eight	nine	cipher, or nought.

6. The art of writing down numbers in figures is called NOTATION; the art of expressing numbers in words (spoken or written), NUMERATION.

7. The largest number that can be expressed by means of one figure is *nine*. Every larger number is represented by a combination of two or more figures.

* In this system we reckon by *tens*: hence the name DECIMAL—from *decem*, Latin for "ten." The system was introduced into Europe by the Arabs, about 800 years ago: hence the name ARABIC. The Hindoos, however, are the people with whom the system is supposed to have originated.

In order to understand the peculiar principle upon which larger numbers than nine are expressed in figures, let us suppose the case of a teacher who employs a class of boys to keep an account, with their fingers, of the number of nuts which he removes, one by one, from a heap before him—the first boy putting up a finger for every nut removed, and beginning again when his fingers are all up; the second boy putting up a finger for every occasion on which the first boy is obliged to begin again; the third boy putting up a finger for every occasion on which the second boy—*his* fingers being all up—is obliged to begin again; and so on. It is evident that, according to this arrangement, every finger held up by the first boy would represent *one* nut; every finger held up by the second boy, *ten* nuts (or as many as the first boy's ten fingers); every finger held up by the third boy, *one hundred* nuts (or as many as the second boy's ten fingers); &c. So that—to take a particular case—*four* of the first boy's fingers, *three* of the second boy's, and *two* of the third boy's would represent, respectively, *four* nuts, *thirty* nuts, and *two hundred* nuts; altogether, *two hundred and thirty-four.* The following is the way in which this number would be expressed if every finger were represented by a "stroke"—the boys being supposed to stand in a row, and in the order indicated:

//	///	////
(3rd boy)	(2nd boy)	(1st boy)

If, however, instead of four, three, and two *strokes*, we wrote the *digits* 4, 3, and 2, respectively, the number *two hundred and thirty-four* would appear under this more concise form—

234.

It is easy to see that, upon the principle just explained, these digits, if differently combined, would express a different number. For instance: *three* of the third boy's fingers, *four* of the second boy's, and *two* of the first boy's would represent (altogether) *three hundred and forty-two*, which would be written—

342;

four of the third boy's fingers, *two* of the second boy's, and *three* of the first boy's would represent *four hundred and twenty-three*, which would be written—

423;

&c. Comparing the three combinations 234, 342, and 423, we see that *the same digit has different values in different places*, and that *different digits have different values in the same place.* Thus, 4 represents *four* in the first combination, *forty* in the second, and *four hundred* in the third; 3 represents *thirty* in the first combination, *three hundred* in the second, and *three* in

the third; whilst 2 represents *two hundred* in the first combination, *two* in the second, and *twenty* in the third.

After the removal of *five hundred and sixty* nuts, *five* of the third boy's fingers, *six* of the second boy's, and *none* of the first boy's would be up. This number of nuts would be written—

560:

the cipher, whilst representing no portion of the number, being necessary in the first place, that 6 may stand in the second, and 5 in the third place; just as the first boy, although having no finger up, would be obliged to keep his place, that there may be no mistake as to the places of the other two boys.

8. When a number is represented by a combination of figures, we find what number it is by adding the individual values of the figures together.

Thus, the combination 365 represents *three hundred and sixty-five*—3 representing *three hundred*; 6, *sixty*; and 5, *five*. The combination 708 represents *seven hundred and eight*—the value of 7 being *seven hundred*; (of 0, *nothing*;) and of 8, *eight*.

9. The cipher has no value in any situation: it merely serves, in certain cases (when no other figure would answer), to keep the digits in their proper places.

10. The value of a digit depends—partly upon *what* digit it is, and partly upon the *place* it occupies.*

The mere *name* does not indicate the value of a digit, because, as we have seen, "the same digit has different values in

* In almost every treatise on Arithmetic, we are told that "a digit has *two* values—simple and local." To be convinced of the absurdity of this, we have merely to take any combination of figures—say 789, and reflect whether, in this combination, 9 has any other value than *nine*, or 8 any other value than *eighty*, or 7 any other value than *seven hundred*. Those who employ the terms *simple* and *local* must intend "simple" to mean "*non*-local;" but when has a digit a non-local value? If it be said that the simple or non-local value of the digit 9, for example, is *nine*, the answer is this: 9 must occupy a particular "place" (the *units'* place) to stand for *nine*—just as it must be in a particular place to represent *ninety* or *nine hundred*; so that *nine* is as "local" a value as any other. The truth is that a digit might have *any* number of values, because it could be written in any number of different places, in no two of which would its value be the same. So long, however, as it remains in the same place (whatever the place may be), a digit has one—and *only* one—value.

different places." The digit called "one" (1), for example, sometimes represents *one*, sometimes *ten*, sometimes *one hundred*, &c. Neither does its *place* indicate what a digit stands for, because "different digits have different values in the same place." In the place in which 1 would stand for *one*, 2 would stand for *two*, 3 for *three*, &c.; in the plaee in which 1 would stand for *ten*, 2 would stand for *twenty*, 3 for *thirty*, &c.; in the place where 1 would stand for *one hundred*, 2 would stand for *two hundred*, 3 for *three hundred*, &c.

11. The place in which the digit 1 would represent a *unit* is termed the "units' place." The next place on the left, where 1 would represent *ten* (units), is called the "tens' place." The next place on the left, where 1 would stand for *one hundred* (units), is called the "hundreds' place." [Thus, in the combination 365, the units' place is occupied by 5, the tens' place by 6, and the hundreds' place by 3; because the digit 1 would represent *one* if substituted for 5, *ten* if substituted for 6, and *one hundred* if substituted for 3.] In the places farther to the left, 1 would represent, respectively, *one thousand, ten thousands, one hundred thousands; one million, ten millions, one hundred millions; one billion, ten billions, one hundred billions;* &c. :—

1 *one hundred* BILLIONS
1 *ten* BILLIONS
1 *one* BILLION
1 *one hundred* MILLIONS
1 *ten* MILLIONS
1 *one* MILLION
1 *one hundred* THOUSANDS
1 *ten* THOUSANDS
1 *one* THOUSAND
1 *one hundred* UNITS
1 *ten* UNITS
1 *one* UNIT

12. The first three places (in which "units" occur) constitute what is called the UNITS' "period;" the next three (in which "thousands" occur), the THOUSANDS' period; the next three (in which "millions" occur), the MILLIONS' period; &c.

Farther to the left would come TRILLIONS, QUADRILLIONS, QUINTILLIONS, SEXTILLIONS, SEPTILLIONS, OCTILLIONS, NONIL-

LIONS, &c., respectively: a trillion being a thousand billions; a quadrillion, a thousand trillions; a quintillion, a thousand quadrillions; &c.*

It will be seen that 1 is everywhere read "one," or "ten," or "one hundred": thus—*one* unit, *one* thousand, *one* million, &c.; *ten* units, *ten* thousands, *ten* millions, &c.; *one hundred* units, *one hundred* thousands, *one hundred* millions, &c.

13. As the place in which 1 represents *one* UNIT is termed the *units'* place of the UNITS' period—so, the place in which 1 represents *one* THOUSAND is termed the *units'* place of the THOUSANDS' period; the place in which 1 represents *one* MILLION, the *units'* place of the MILLIONS' period; &c. Again: as the place in which 1 represents *ten* UNITS is termed the *tens'* place of the UNITS' period—so, the place in which 1 represents *ten* THOUSANDS is termed the *tens'* place of the THOUSANDS' period; the place in which 1 represents *ten* MILLIONS, the *tens'* place of the MILLIONS' period; &c. In like manner, as the place in which 1 stands for *one hundred* UNITS is called the *hundreds'* place of the UNITS' period—so, the place in which 1 stands for *one hundred* THOUSANDS is called the *hundreds'* place of the THOUSANDS' period; the place in which 1 stands for *one hundred* MILLIONS, the *hundreds'* place of the MILLIONS' period; &c.:—

hundreds' place	*tens'* place	*units'* place	*hundreds'* place	*tens'* place	*units'* place	*hundreds'* place	*tens'* place	*units'* place	*hundreds'* place	*tens'* place	*units'* place
1	1	1	1	1	1	1	1	1	1	1	1
BILLIONS' period.			MILLIONS' period.			THOUSANDS' period.			UNITS' period.		

* In the English system of notation, a combination of figures would be divided into periods of *six* each—a billion meaning a million of millions; a trillion, a million of billions; a quadrillion, a million of trillions; &c. This system may be said to have been entirely superseded by the one described above, and which is sometimes spoken of, for the sake of distinction, as the French or Continental system of notation.

14. There are places to the right, as well as to the left, of the units' place; so that the most right-hand figure of a combination is not always the units' figure.* Looking at the combination before us, we observe that, as we pass from left to right, the digit 1 becomes continually smaller in value, until it represents only a unit. We observe, moreover, that the digit becomes *ten* times less in value when removed *one* place to the right, *one hundred* times less when removed *two* places to the right, *one thousand* times less when removed *three* places to the right, and so on.† Bearing this in mind, we are prepared to be told that, in any of the places to the right of the units' place, 1 would be less in value than—or would represent only *part* of—a unit; and further, that, in the place immediately to the right of the units' place, 1 would be *ten* times less in value than—or would represent the *tenth* part of—a unit; that, in the next place on the right, 1 would be a *hundred* times less in value than—or would represent the *hundredth* part of—a unit; that, in the next place on the right, 1 would be a *thousand* times less in value than—or would represent the *thousandth* part of—a unit; &c.:—

one UNIT		one *tenth*	one *hundredth*	one *thousandth*	one *ten-thousandth*	one *hundred-thousandth*	one *millionth*	one *ten-millionth*	one *hundred-millionth*	one *billionth*
1	.	1	1	1	1	1	1	1	1	1

* When we speak of the "units' place," or the "units' figure," without mentioning any period in particular, the UNITS' period is understood to be the one referred to.

† This is merely another way of saying that the digit becomes ten times *greater* in value when removed one place to the *left*, one hundred times greater when removed two places to the left, one thousand times greater when removed three places to the left, &c.

15. As the values of the first three figures to the left of the units' period are expressed in *thousands*; of the next three, in *millions*; of the next three, in *billions*; &c.—so, the values of the first three figures to the right of the units' period are expressed in *thousandths*; of the next three, in *millionths*; of the next three, in *billionths*; &c. Let us re-write the last combination, omitting the names which we intend to dispense with, and observing where 1 represents, respectively, one *thousandth*, one *millionth*, and one *billionth*:—

one UNIT			one *thousandth*			one *millionth*			one *billionth*
1 ·	1	1	1	1	1	1	1	1	1
	(*a*)	(*b*)	(*c*)	(*d*)	(*e*)	(*f*)	(*g*)	(*h*)	(*k*)

Now, if we remember that 1 becomes ten times greater in value when removed one place, and a hundred times greater when removed two places to the left, it will be evident that the number of THOUSANDTHS which the digit represents in the place marked (*b*) is *ten*, and in the place marked (*a*) *one hundred*—the number represented in the place marked (*c*) being *one*; that the number of MILLIONTHS represented in the place marked (*e*) is *ten*, and in the place marked (*d*) *one hundred*—the number represented in the place marked (*f*) being *one*; and that the number of BILLIONTHS represented in the place marked (*h*) is *ten*, and in the place marked (*g*) *one hundred*—the number represented in the place marked (*k*) being *one*:—

one UNIT		*one hundred* THOUSANDTHS	*ten* THOUSANDTHS	*one* THOUSANDTH	*one hundred* MILLIONTHS	*ten* MILLIONTHS	*one* MILLIONTH	*one hundred* BILLIONTHS	*ten* BILLIONTHS	*one* BILLIONTH
1	.	1	1	1	1	1	1	1	1	1

So that to the right, as well as to the left, of the units' place, 1 is everywhere read "one," or "ten," or "one hundred": thus—*one* thousandth, *one* millionth, *one* billionth, &c.; *ten* thousandths, *ten* millionths, *ten* billionths, &c.; *one hundred* thousandths, *one hundred* millionths, *one hundred* billionths, &c.

16. The place in which 1 represents *one* THOUSANDTH is called the *units'* place of the THOUSANDTHS' period; the place in which 1 represents *one* MILLIONTH, the *units'* place of the MILLIONTHS' period; the place in which 1 represents *one* BILLIONTH, the *units'* place of the BILLIONTHS' period; &c. The place in which 1 represents *ten* THOUSANDTHS is called the *tens'* place of the THOUSANDTHS' period; the place in which 1 represents *ten* MILLIONTHS, the *tens'* place of the MILLIONTHS' period; the place in which 1 represents *ten* BILLIONTHS, the *tens'* place of the BILLIONTHS' period; &c. The place in which 1 stands for *one hundred* THOUSANDTHS is termed the *hundreds'* place of the THOUSANDTHS' period; the place in which 1 stands for *one hundred* MILLIONTHS, the *hundreds'* place of the MILLIONTHS' period; the place in which 1 stands for *one hundred* BILLIONTHS, the *hundreds'* place of the BILLIONTHS' period; &c.:—

one UNIT		*hundreds'* place	*tens'* place	*units'* place	*hundreds'* place	*tens'* place	*units'* place	*hundreds'* place	*tens'* place	*units'* place
1	.	1	1	1	1	1	1	1	1	1
		THOUSANDTHS' period.			MILLIONTHS' period.			BILLIONTHS' period.		

17. We see, then, that every other period, as well as the UNITS' period, has a *units'*, a *tens'*, and a *hundreds'* place; that the places of every other period occupy the same *relative* positions as those of the UNITS' period—the units' place being invariably on the right, the hundreds' place on the left, and the tens' place in the middle; and that, immediately to the left of the hundreds' place of one period, is found the units' place of the next higher period—in other words, that, immediately to the right of the units' place of one period, is found the hundreds' place of the next lower period. So that, in every situation, a figure occupies the units', or the tens', or the hundreds' place of *some* period :—

hundreds' place	*tens'* place	*units'* place	*hundreds'* place	*tens'* place	*units'* place	*hundreds'* place	*tens'* place	*units'* place		*hundreds'* place	*tens'* place	*units'* place	*hundreds'* place	*tens'* place	*units'* place
1	1	1	1	1	1	1	1	1	·	1	1	1	1	1	1
MILLIONS' period.			THOUSANDS' period.			UNITS' period.				THOUSANDTHS' period.			MILLIONTHS' period.		

18. The *units'* place (of the UNITS' period) has a point, called the DECIMAL POINT, immediately to the right of it. When, however, no figure occurs farther to the right, the point is left unwritten. So that in every case in which the decimal point is written, the figure immediately to the left of it is the units' figure; whilst in every case in which the point is not written, the *most right-hand* figure is the units' figure.

Thus, the units' place is occupied by 4 in the first of the following combinations, by 3 in the second, and by 2 in the third; whilst, in the fourth combination, the units' place is unoccupied, being the place immediately to the left of that occupied by the digit 2* :—

234; 23·4; 2·34; ·234.

---

* It will be well to remember that the decimal point does *not* occupy a "*place*."

19. Whatever the digit 1 would represent in any place, 2 in that place would represent *twice* as much; 3, *three* times as much; 4, *four* times as much; &c.

The following are illustrations :—

100, 100, 100·100, 1$\cancel{0}\cancel{0}$	*one hundred* MILLIONS	*one hundred* THOUSANDS	*one hundred* UNITS	*one hundred* THOUSANDTHS	*one hundred* MILLIONTHS
200, 200, 200·200, 2$\cancel{0}\cancel{0}$	*two hundred* MILLIONS	*two hundred* THOUSANDS	*two hundred* UNITS	*two hundred* THOUSANDTHS	*two hundred* MILLIONTHS
300, 300, 300·300, 3$\cancel{0}\cancel{0}$	*three hundred* MILLIONS	*three hundred* THOUSANDS	*three hundred* UNITS	*three hundred* THOUSANDTHS	*three hundred* MILLIONTHS
&c.					
$\cancel{0}$10, 010, 010·010, 01$\cancel{0}$	*ten* MILLIONS	*ten* THOUSANDS	*ten* UNITS	*ten* THOUSANDTHS	*ten* MILLIONTHS
$\cancel{0}$20, 020, 020·020, 02$\cancel{0}$	*twenty* MILLIONS	*twenty* THOUSANDS	*twenty* UNITS	*twenty* THOUSANDTHS	*twenty* MILLIONTHS
$\cancel{0}$30, 030, 030·030, 03$\cancel{0}$	*thirty* MILLIONS	*thirty* THOUSANDS	*thirty* UNITS	*thirty* THOUSANDTHS	*thirty* MILLIONTHS
&c.					
$\cancel{0}\cancel{0}$1, 001, 001·001, 001	*one* MILLION	*one* THOUSAND	*one* UNIT	*one* THOUSANDTH	*one* MILLIONTH
$\cancel{0}\cancel{0}$2, 002, 002·002, 002	*two* MILLIONS	*two* THOUSANDS	*two* UNITS	*two* THOUSANDTHS	*two* MILLIONTHS
$\cancel{0}\cancel{0}$3, 003, 003·003, 003	*three* MILLIONS	*three* THOUSANDS	*three* UNITS	*three* THOUSANDTHS	*three* MILLIONTHS
&c.					

Here are additional illustrations :—

230,	*two hundred and thirty* MILLIONS
230,	*two hundred and thirty* THOUSANDS
230·	*two hundred and thirty* UNITS
230,	*two hundred and thirty* THOUSANDTHS
23~~0~~	*two hundred and thirty* MILLIONTHS

456,	*four hundred and fifty-six* MILLIONS
456,	*four hundred and fifty-six* THOUSANDS
456·	*four hundred and fifty-six* UNITS.
456,	*four hundred and fifty-six* THOUSANDTHS
456	*four hundred and fifty-six* MILLIONTHS

&c.

~~0~~23,	*twenty-three* MILLIONS
023,	*twenty-three* THOUSANDS
023·	*twenty-three* UNITS
023,	*twenty-three* THOUSANDTHS
023	*twenty-three* MILLIONTHS

302,	*three hundred and two* MILLIONS
302,	*three hundred and two* THOUSANDS
302·	*three hundred and two* UNITS
302,	*three hundred and two* THOUSANDTHS
302	*three hundred and two* MILLIONTHS

&c.

From the preceding illustrations, in which, however, the cancelled ciphers are unnecessary (see p. 13), we come to this conclusion:

20. In expressing any number of "thousands," "millions," "thousandths," or "millionths," &c., we employ the same digit or digits—and in the same relative positions, when there are more digits than one—as if we wanted to write so many "units." No exercise in notation, therefore, ought to present

the slightest difficulty to those who are able to express any number of "units" that may be mentioned, and who know how the several periods are situated with respect to one another. For, in writing a number, the pupil has merely to proceed period by period; to consider, in each case, what digit or digits must be employed, and in what place or places of the period; and to fill up the intermediate places with ciphers—a comma being written, as a point of separation, after every period but the last, except in the case of the UNITS' period, which will be sufficiently separated from the THOUSANDTHS' period by the decimal point.

It is to be observed that whilst commas are always written at the bottom, the decimal point is as invariably written higher up—sometimes half-way up (after the *middle* of the units' figure), and sometimes at the top. It may also be observed that the cipher, when it occurs in a combination of figures, is (or ought to be) situated either between two digits or between a digit and the decimal point: because in no other position is the cipher of any use.*

EXERCISES IN NOTATION AND NUMERATION.

1. Express in figures *three hundred* MILLIONS *forty* THOUSANDS *and five* (UNITS).†

Three hundred MILLIONS will be expressed by the digit 3 in the *hundreds'* place of the MILLIONS' period; *forty* THOUSANDS by 4 in the *tens'* place of the THOUSANDS' period; and *five* (UNITS) by 5 in the *units'* place of the UNITS' period. We there-

---

* If we set down the following combinations, and remove the ciphers, no one of which occurs either between two digits or between a digit and the decimal point, the values of the combinations will remain unaltered; because the place (and therefore the value) of each digit will be the same after, as before, the removal:—

$\cancel{0}234$ $\qquad \cdot 56\cancel{0}$ $\qquad \cancel{0}78\cdot 9\cancel{0}$

Here, on the other hand, are combinations whose values would be altered by the removal of the ciphers, situated, as each cipher is, either between two digits or between a digit and the decimal point:—

830 $\qquad$ ·107 $\qquad$ 205·049

The removal of the ciphers would change the first of these combinations into 83, the second into ·17, and the third into 25·49—the digits 8 and 3 being each removed a place to the right, 7 a place to the left, 2 a place to the right, and 4 and 9 a place each to the left.

† In practice, we say "million" instead of "millions;" "thousand" instead of "thousands;" &c.

fore begin by writing 3, 0, 0, as the *hundreds'*, the *tens'*, and the *units'* figure, respectively, of the MILLIONS' period; we next write 0, 4, 0, as the *hundreds'*, the *tens'*, and the *units'* figure, respectively, of the THOUSANDS' period; and we finish by writing 0, 0, 5, as the *hundreds'*, the *tens'*, and the *units'* figure, respectively, of the UNITS' period—remembering that, as we pass from left to right, the order of the places of a period is invariably this:—(*a*) *hundreds'* place, (*b*) *tens'* place, (*c*) *units'* place. The work in its different stages is shown in the margin, the end of one period being marked with a comma before we pass to the next period on the right.

300,
300,040,
300,040,005

2. Express in figures *sixty* THOUSANDTHS *seven* MILLIONTHS *and eight hundred* BILLIONTHS.

In this case we begin by setting down the *decimal point*, as all the figures are to be to the right of it. We then write 0, 6, 0, as the *hundreds'*, the *tens'*, and the *units'* figure, respectively, of the THOUSANDTHS' period; 0, 0, 7, as the *hundreds'*, the *tens'*, and the *units'* figure, respectively, of the MILLIONTHS' period; and 8, 0, 0, as the *hundreds'*, the *tens'*, and the *units'* figure, respectively, of the BILLIONTHS' period (the last two figures, however, are unnecessary)—knowing that *sixty* THOUSANDTHS will be expressed by 6 in the *tens'* place of the THOUSANDTHS' period, *seven* MILLIONTHS by 7 in the *units'* place of the MILLIONTHS' period, and *eight hundred* BILLIONTHS by 8 in the *hundreds'* place of the BILLIONTHS' period.

·060,
·060,007,
·060,007,8~~00~~

3. Express in figures *eighty-seven* BILLIONS *three hundred and sixty-five* MILLIONS *four hundred and ninety* UNITS *six hundred and two* THOUSANDTHS *and nine hundred and ten* BILLIONTHS.

Eighty-seven BILLIONS will be expressed by the digits 8 and 7 in the *tens'* and the *units'* place, respectively, of the BILLIONS' period; *three hundred and sixty-five* MILLIONS by 3, 6, and 5, in the *hundreds'*, the *tens'*, and the *units'* place, respectively, of the MILLIONS' period; *four hundred and ninety* UNITS by 4 and 9 in the *hundreds'* and the *tens'* place, respectively, of the UNITS' period; *six hundred and two* THOUSANDTHS by 6 and 2 in the *hundreds'* and the *units'* place, respectively, of the THOUSANDTHS' period; and *nine hundred and ten* BILLIONTHS by 9 and 1 in the *hundreds'* and the *tens'* place, respectively, of the BILLIONTHS'

87,
87,365,
87,365,000,
87,365,000,490·
87,365,000,490·602,
87,365,000,490·602,000,
87,365,000,490·602,000,91~~0~~

period. We therefore proceed as shown in the margin, taking care, in passing from the MILLIONS' to the UNITS' period, to fill up the places of the THOUSANDS' period with ciphers (there being no thousands); and taking care, also, in passing from the THOUSANDTHS' to the BILLIONTHS' period, to fill up the places of the MILLIONTHS' period with ciphers (there being no millionths).

[When we want to read a number expressed in figures, we first divide the figures—if they be not already divided—into periods, always commencing at the *units'* place (of the UNITS' period). We then see, at a glance, what periods there are; also, what places of each are occupied by digits, and by what digits. In fact, as we pass from left to right, each period, in turn, is read as if it were the UNITS' period, except that, instead of "units," we say "millions" in the case of the MILLIONS' period, "thousands" in the case of the THOUSANDS' period, &c.]

4. Express in words the value of the combination 52647030

Dividing the figures into periods, we see that this number consists of MILLIONS, THOUSANDS, and UNITS:

52,647,030

We see, moreover, that 5 and 2 are the *tens'* and the *units'* figure, respectively, of the MILLIONS' period; that 6, 4, and 7 are the *hundreds'*, the *tens'*, and the *units'* figure, respectively, of the THOUSANDS' period; and that 0, 3, 0 are the *hundreds'*, the *tens'*, and the *units'* figure, respectively, of the UNITS' period. The most left-hand period, therefore, is read *fifty-two* MILLIONS—just as, if it were the UNITS' period, it would be read *fifty-two* (UNITS); and the middle period is read *six hundred and forty-seven* THOUSANDS—just as, if it were the UNITS' period, it would be read *six hundred and forty-seven* (UNITS). We therefore express the given numbers in words by saying—

Fifty-two MILLIONS *six hundred and forty-seven* THOUSANDS *and thirty* (UNITS).

5. Express in words the value of the combination ·2197054

Dividing the figures into periods, we see that this number consists of THOUSANDTHS, MILLIONTHS, and BILLIONTHS:

·219,705,4

We also see that the digits 2, 1, 9 occupy the *hundreds'*, the *tens'*, and the *units'* place, respectively, of the THOUSANDTHS' period; that 7 and 5 occupy the *hundreds'* and the *units'* place, respectively, of the MILLIONTHS' period; and that 4 (being immediately to the right of the *units'* place of the THOUSANDTHS' period) occupies the *hundreds'* place of the BILLIONTHS' period.

So that the number of THOUSANDTHS is *two hundred and nineteen;* of MILLIONTHS, *seven hundred and five;* and of BILLIONTHS, *four hundred.* The given number is therefore read—

Two hundred and nineteen THOUSANDTHS *seven hundred and five* MILLIONTHS *and four hundred* BILLIONTHS.

6. Express in words the value of the combination 6000801534·00002795

The figures being divided into periods, this number is found to consist of BILLIONS, THOUSANDS, UNITS, MILLIONTHS, and BILLIONTHS:*

6,000,801,534·000,027,95

It is also found that 6 (being immediately to the left of the *hundreds'* figure of the MILLIONS' period) is the *units'* figure of the BILLIONS' period; that 8 and 1 are the *hundreds'* and the *units'* figure, respectively, of the THOUSANDS' period; that 5, 3, and 4 are the *hundreds'*, the *tens'*, and the *units'* figure, respectively, of the UNITS' period; that 2 and 7 are the *tens'* and the *units'* figure, respectively, of the MILLIONTHS' period; and that 9 and 5 are the *hundreds'* and the *tens'* figure, respectively, of the BILLIONTHS' period. So that the number of BILLIONS is *six;* of THOUSANDS, *eight hundred and one;* of UNITS, *five hundred and thirty-four;* of MILLIONTHS, *twenty-seven;* and of BILLIONTHS, *nine hundred and fifty.* We therefore read the given number—

Six BILLIONS *eight hundred and one* THOUSANDS *five hundred and thirty-four* UNITS *twenty-seven* MILLIONTHS *and nine hundred and fifty* BILLIONTHS.

From an examination of such combinations as—

2222222·222222
3333333·333333
4444444·444444
&c.

it will be seen that (not merely the digit 1, already referred to, but) *any* digit becomes GREATER in value when removed to the LEFT, and LESS when removed to the RIGHT:

$$\left.\begin{array}{r}10\\100\\1{,}000\\10{,}000\\100{,}000\\1{,}000{,}000\end{array}\right\}\text{ times greater or less (as the case may be) when removed }\left\{\begin{array}{l}1\text{ place}\\2\text{ places}\\3\quad,,\\4\quad,,\\5\quad,,\\6\quad,,\end{array}\right.$$
&c. &c.

* There are no MILLIONS or THOUSANDTHS, none of the places of either of these periods being occupied by a digit.

This being so, it is obvious that any combination of figures will become 10 times greater in value if each figure be removed one place to the left, 100 times greater if each figure be removed two places to the left, 1,000 times greater if each figure be removed three places to the left, &c.; also, that any combination will become 10 times less in value if each figure be removed one place to the right, 100 times less if each figure be removed two places to the right, 1,000 times less if each figure be removed three places to the right, &c. Thus, by removing every figure of the combination 456·789 one place to the left—9 into the place occupied by 8, 8 into the place occupied by 7, 7 into the place occupied by 6, &c.—we make the value of each digit, and therefore of all taken together (that is, the value of the combination) 10 times greater:

456·789
(*a*) 4567·89

By removing every figure two places to the left—9 into the place occupied by 7, 8 into the place occupied by 6, 7 into the place occupied by 5, &c.—we make the value of each digit, and therefore of all taken together, 100 times greater:—

456·789
(*b*) 45678·9

By removing every figure three places to the left—9 into the place occupied by 6, 8 into the place occupied by 5, 7 into the place occupied by 4, &c.—we make the value of each digit, and therefore of all taken together, 1,000 times greater:—

456·789
(*c*) 456789

On the other hand, by removing every figure one place to the right—4 into the place occupied by 5, 5 into the place occupied by 6, 6 into the place occupied by 7, &c.—we make each digit, and therefore all taken together, 10 times less in value:—

456·789
(*d*) 45·6789

By removing every figure two places to the right—4 into the place occupied by 6, 5 into the place occupied by 7, 6 into the place occupied by 8, &c.—we make each digit, and therefore all taken together, 100 times less in value:—

456·789
(*e*) 4·56789

By removing every figure three places to the right—4 into the place occupied by 7, 5 into the place occupied by 8, 6 into the

place occupied by 9, &c.—we make each digit, and therefore all taken together, 1,000 times less in value:—

456·789
(*f*) ·456789

Without actually removing the figures, we are able to effect the preceding changes by merely altering the position of the decimal point. Thus, leaving the figures as they are, we can convert 456·789 (the original combination) into the combination marked (*a*), 10 times greater in value, by removing the point one place to the right; into the combination marked (*b*), 100 times greater, by removing the point two places to the right; into the combination marked (*c*), 1,000 times greater, by removing the point three places to the right: into the combination marked (*d*), 10 times less in value, by removing the point one place to the left; into the combination marked (*e*), 100 times less, by removing the point two places to the left; and into the combination marked (*f*), 1,000 times less, by removing the point three places to the left. The following conclusion is thus arrived at:

21. Whenever the position of the decimal point is altered, the value of a combination is made greater or less—GREATER when the point is removed to the RIGHT, and LESS when the point is removed to the LEFT; and the number denoting *how many times* greater or less the value of the combination becomes in such cases is either 10 or one of the numbers (100 — 1,000 — 10,000 — 100,000 — 1,000,000 — &c.) called *powers* of 10: in other words, is always expressed by the digit 1 followed by one or more ciphers—the *number of ciphers* being invariably the same as the *number of places* the point is removed.

EXERCISES ON THE CHANGES PRODUCED BY THE REMOVAL OF THE DECIMAL POINT.

I. Convert 35 into a combination 10 times greater in value.

In writing 10, we employ *one* cipher. We therefore remove the decimal point* of the given combination *one* place to the

* It may not be unnecessary to repeat that the decimal point of a combination, when not written, is always understood to be immediately to the right of the most right-hand figure, which, in such cases, is the units' figure.

right, and this we do (in the absence of a figure to the right of 5) by annexing a cipher, and making it the units' figure: 350

II. Convert ·7 into a number 100 times smaller.

We effect this change by removing the decimal point *two* places to the left—*two* ciphers being employed in the writing of 100; and as there is no figure to the left of the point, we prefix two ciphers, before the left-hand one of which we write the point: ·007

III. Convert 4·68 into a number 1,000 times larger.

In this case we remove the decimal point *three* places to the right—*three* ciphers being employed in the writing of 1,000; and as there are only two figures to the right of the point, we annex one cipher, which we make the units' figure: 4680

IV. How many times smaller is ·0009 than 9?

If its decimal point were removed *four* places to the right, the combination ·0009 would become 9. Writing *four* ciphers after the digit 1, therefore, we see that 9 is 10,000 times a larger number than ·0009; in other words, that ·0009 is 10,000 times smaller than 9.

V. How many times larger is 30,000 than ·3?

If its decimal point were removed *five* places to the left, the combination 30,000 would become ·3. Writing *five* ciphers after 1, therefore, we see that ·3 is 100,000 times smaller than 30,000; in other words, that 30,000 is 100,000 times larger than ·3.

VI. How many times smaller is ·0864312 than 86431·2?

If its decimal point were removed *six* places to the right, the combination ·0864312 would become 086431·2. Writing *six* ciphers after 1, therefore, we see that 86431·2 is 1,000,000 times larger than ·0864312; in other words, that ·0864312 is 1,000,000 times smaller than 86431·2.

NOTE.—By remembering that, in a certain sense, his right hand is the "larger," and his left hand the "smaller" of the two, the pupil will have no difficulty in determining, in such

cases as the preceding, when the decimal point should be removed to the right, and when to the left. Thus, when the point is removed to the *right*—that is, towards the "larger" hand—the number represented by the combination becomes *larger*; whilst, when the point is removed to the *left*—that is, towards the "smaller" hand—the value of the combination becomes *smaller*.

The following illustrations explain how, in performing arithmetical operations, when the figures of a combination have to be dealt with individually (instead of in periods, or altogether), we are able—whilst expressing its value—to call each digit by its *name*:—

3 *three* groups of 1,000 each	2 *two* groups of 1,000 each	1 *one* group of 1,000
3 *three* groups of 100 each	2 *two* groups of 100 each	1 *one* group of 100
3 *three* groups of 10 each	2 *two* groups of 10 each	1 *one* group of 10
3· *three* units	2· *two* units	1· *one* unit
3 *three* tenths	2 *two* tenths	1 *one* tenth
3 *three* hundredths	2 *two* hundredths	1 *one* hundredth
3 *three* thousandths	2 *two* thousandths	1 *one* thousandth
&c.	&c.	&c.

22. A number expressed by one or more figures to the right of the decimal point is called a DECIMAL, and the places to the right of the point are usually spoken of as DECIMAL PLACES. A number expressed by one or more figures to the left of the decimal point is termed an INTEGER or a WHOLE NUMBER.

Although, in a comprehensive examination of the Decimal System of Notation, we cannot avoid referring to the millionths' and the billionths' period, it is hardly ever necessary, in practice, to go farther to the right of the decimal point than two or three places. This will be evident when it is remembered that, if the unit under consideration were so large a sum of money as a pound, the digit 1 in the third decimal place (£·001) would represent only the thousandth part of a pound—that is, would represent *less than a farthing*. Such decimals as occur in practice are commonly read in this way:—

·304 = "*Point—three—nought—four.*"
·056 = "*Point—nought—five—six.*"
·2708 = "*Point—two—seven—nought—eight.*"
&c. &c.

ROMAN SYSTEM OF NOTATION.

23. The system of notation employed by the ancient Romans, although very imperfect, and (except in the case of such matters as dates) of scarcely any practical importance at the present day, is supposed to be familiar to every arithmetician. In this system, which exhibits unmistakable evidence of an attempt at a decimal scale, the numbers 1, 10, 100, and 1,000 are represented by the letters I, X, C, and M, respectively; and 5, 50, and 500 by V, L, and D, respectively. Moreover, the numbers 4, 40, and 400 are represented by the combinations IV, XL, and CD, respectively; whilst 9, 90, and 900 are represented by IX, XC, and CM, respectively. Those of the six combinations just mentioned are the only cases in which a character is placed before one of greater value; and it will be observed that each combination represents the *difference* between the individual values of the two characters composing it. Thus, IV represents the difference between the value of I (1) and that of V (5); XL the difference between the value of X (10) and that of L (50); CD the difference between the value of C (100) and that of D (500); &c. For convenience' sake, we shall speak of I, V, X, L, C, D, and M as "the seven *single* characters;" and of IV, IX, XL, XC, CD, and CM (not as "combinations," but) as "the six *double* characters:"—

*Single Characters.**	*Double Characters.**
I= 1	IV= 4
V= 5	IX= 9
X= 10	XL= 40
L= 50	XC= 90
C= 100	CD=400
D=500	CM=900
M=1,000	

* Those who may be disposed to find fault with the expressions "Single" and "*Double*," as employed here, would do well to remember

24. To find the value of a combination of Roman characters, we *add the individual values of the characters together.*

EXAMPLE I.—Find the value of XVI.

Of the three ("single") characters in this combination, the first (X) represents 10, the second (V) 5, and the third (I) 1: the value of the combination, therefore, is 16—the sum of 10, 5, and 1.

EXAMPLE II.—Find the value of CMXLIX.

Of the three ("double") characters in this combination, the first (CM) represents 900, the second (XL) 40, and the third (IX) 9: the value of the combination, therefore, is 949.

EXAMPLE III.—Find the value of CDXXXIV.

In this case, there are two "double" characters (CD and IV) and a "single" one (X)—the single character occurring three times. The first character (CD) represents 400, and the next three (XXX), taken together, 30—that is, 10 each, whilst the value of the last character (IV) is 4: the value of the combination, therefore, is 434.

25. By placing a bar (—) over it, we make a character or combination of characters 1,000 times greater in value. Thus:

$\overline{X}$ =	10,000	$\overline{IV}$ =	4,000	$\overline{XV}$ =	15,000
$\overline{C}$ =	100,000	$\overline{XL}$ =	40,000	$\overline{CXX}$ =	120,000
$\overline{M}$ =	1,000,000	$\overline{CD}$ =	400,000	$\overline{MDCCC}$ =	1,800,000

The bar is supposed to be employed, however, only when, by means of it, we are able to write a number with fewer characters than would otherwise be necessary. For instance: we prefer M to $\overline{I}$ for 1,000, MM to $\overline{II}$ for 2,000, and MMM to $\overline{III}$ for 3,000;

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that in no other treatise on Arithmetic—so far as the Author has been able to discover—is any explanation given of the Roman system, beyond this: that there are seven characters; that, when a character is placed before a higher one, the value of the former is to be taken from that of the latter; and that, when a character is placed after a higher one, the values of the two are to be added together. This explanation, which would seem to imply that *more than two* characters never occur in a combination, leaves the pupil under the impression that, in expressing a number, he is at liberty to combine the characters in any way he pleases; that, *for example, he can* write IC for 99, VD for 495, LM for 950, &c.

but, on the other hand, instead of MMMM, for 4,000, we write $\overline{IV}$.

26. Rule for the writing of a number in Roman characters: 1. Set down the highest character (single or double, and with or without the bar, as the case may be) whose value does not exceed the given number. 2. Then, consider what portion of the number remains to be expressed, and set down the highest character whose value does not exceed this portion. 3. Next, consider how much of the number still remains unwritten, and proceed as before. 4. Continue the process—taking care to annex each character to the one last written—until the whole of the number shall have been set down.

EXAMPLE I.—Express 99 in Roman characters.

We begin by setting down

XC,

the highest character whose value does not exceed 99. Only 9 then remains to be expressed, the value of XC being 90. As 9 will be represented by IX, we annex this character to XC, and thus form the required combination:

XCIX

EXAMPLE II.—Express 1,464 in Roman characters.

We first set down

M,

the highest character whose value does not exceed 1,464. Only 464 then remains to be expressed, the value of M being 1,000. The highest character whose value does not exceed 464 is CD, which, therefore, we annex to M:

MCD

Only 64 then remains to be expressed, the value of CD being 400. The highest character whose value does not exceed 64 is L, which, therefore, we annex to CD:

MCDL

Only 14 then remains to be expressed, the value of L being 50. The highest character whose value does not exceed 14 is X, which, therefore, we annex to L:

MCDLX

The unwritten portion of the number is then reduced to 4, X representing 10. As 4 will be represented by IV, we complete the combination by annexing this character to X:

MCDLXIV

EXAMPLE III.—Express 1,900,305 in Roman characters.

Proceeding as directed by the rule, we first set down

$\overline{\text{M}}$

(for 1,000,000), leaving 900,305 unwritten. We next set down $\overline{\text{CM}}$ (for 900,000), leaving 305 unwritten:

$\overline{\text{MCM}}$

We then set down C (for 100), leaving 205 unwritten:

$\overline{\text{MCM}}$C

Then, another C, leaving 105 unwritten:

$\overline{\text{MCM}}$CC

Then, a third C, leaving only 5 unwritten:

$\overline{\text{MCM}}$CCC

And we complete the combination by writing V for 5:

$\overline{\text{MCM}}$CCCV

27. A double character never repeats itself in a properly-formed combination, and neither does any one of the single characters V (5), L (50), D (500). The reason is obvious. Instead of

IVIV	(for 8)	we write	VIII,
IXIX	(„ 18)	„	XVIII,
XLXL	(„ 80)	„	LXXX,
&c.			&c.;

because the rule obliges us in every case to begin with the *highest* character whose value does not exceed the number to be expressed. Again: instead of

VV	(for 10)	we write	X,
VVV	(„ 15)	„	XV,
LL	(„ 100)	„	C,
LLL	(„ 150)	„	CL,
DD	(„ 1,000)	„	M,
DDD	(„ 1,500)	„	MD,
&c.			&c.

28. Omitting V, L, and D, we have four single characters remaining—I (1), X (10), C (100), and M (1000). Of these four, no one (its appearance as part of a double character not being taken into account) requires to be employed more than *three* times in any combination. Thus, instead of—

IIII*	(for	4)	we write	IV,
VIIII	(„	9)	„	IX,
XXXX	(„	40)	„	XL,
LXXXX	(„	90)	„	XC,
CCCC	(„	400)	„	CD,
DCCCC	(„	900)	„	CM,
MMMM	(„	4,000)	„	$\overline{\text{IV}}$,
&c.				&c.

29. The great imperfection of the Roman system of notation is, that *the same character has always the same value*, being in no way affected by change of position. Thus, whilst the Decimal combination 111 represents *one hundred and eleven*,—the individual values of the figures (as we pass from right to left) being *one*, *ten*, and *one hundred*, respectively,—the value of the Roman combination III is only *three*, each I representing *one*.

NOTE.—The characters D and M are sometimes, although very seldom, met with under the forms IↃ and CIↃ, respectively. When these forms were employed, every Ↄ annexed to IↃ had the effect of making the number 10 times larger; whilst every C and Ↄ joined—the former prefixed and the latter affixed—to CIↃ had (both together) the same effect. Thus—

IↃ	=	500	CIↃ	=	1,000
IↃↃ	=	5,000	CCIↃↃ	=	10,000
IↃↃↃ	=	50,000	CCCIↃↃↃ	=	100,000
IↃↃↃↃ	=	500,000	CCCCIↃↃↃↃ	=	1,000,000
&c.		&c.	&c.		&c.

SIMPLE, COMPOUND, AND FRACTIONAL NUMBERS.

30. There are three classes of numbers—*Simple*, *Compound*, and *Fractional*.

31. A "Simple" number is expressed either by one figure, or by a combination of figures which occupy *consecutive* places in the decimal scale of notation. The following are simple numbers:

9 ·6 807 ·45 23·014

* On the dials of clocks and watches the number 4 is commonly written IIII, but it is so written as a matter of custom or choice—not as a matter of necessity.

32. A "Compound" number consists of two or more simple numbers of the same kind, but of different denominations—written one after the other. The following are compound numbers*:

cwt.	*qrs.*	£	*s.*	*d.*	*miles*	*fur.*	*per.*	*yds.*
10	3	4	17	9	28	6	34	5

33. Respecting a "Fractional" number it will, for the present, be sufficient to say that such a number is expressed by two simple numbers—placed one below the other, and separated by a horizontal line. The following are fractional numbers:

$$\frac{3}{5} \qquad \frac{4}{4} \qquad \frac{8}{3}$$

NOTE.—A number is called Simple, Compound, or Fractional, not on account of its largeness or smallness, but because of the way in which it is expressed. Thus, of the following four numbers, which all represent the *same* amount of money, the first two are simple, whilst the third is compound, and the fourth fractional:

30*d.* £·125 2*s.* 6*d.* £$\frac{1}{8}$

THE FOUR OPERATIONS.

34. All arithmetical operations are reducible to four, namely—ADDITION, SUBTRACTION, MULTIPLICATION, and DIVISION.

35. The rules for the performance of those operations are called—in the case of *simple* numbers, the SIMPLE RULES; in the case of *compound* numbers, the COMPOUND RULES; and in the case of *fractional* numbers, the FRACTIONAL RULES.

36. The symbol

+ (called *plus*) is employed in Addition, to indicate that the two numbers between which it is placed are to be added together;

* It will be seen that the figures of a compound number do *not* occupy "consecutive" places. For, if they did, the figures of the number £4 17*s.* 9*d.*, for example, would represent, respectively, 9 pence, 70 pence, 100 pence, and 4,000 pence, and—if a penny were recognized as the unit—would be written 4179*d.*

$-$ (called *minus*) is employed in Subtraction, to indicate that, of the two numbers between which it is placed, the second is to be taken from the first;

$\times$ (called *multiplied by*) is employed in Multiplication, to indicate that the two numbers between which it is placed are to be multiplied—the one by the other;

$\div$ (called *divided by*) is employed in Division, to indicate that, of the two numbers between which it is placed, the first is to be divided by the second;

$=$ (called *equal to*) is employed to indicate that the two numbers between which it is placed are equal.

Thus, the expression—

$12+3=15$ is read 12 *plus* 3 *equal to* 15;
$12-3=9$ „ „ 12 *minus* 3 *equal to* 9;
$12\times3=36$ „ „ 12 *multiplied by* 3 *equal to* 36;
$12\div3=4$ „ „ 12 *divided by* 3 *equal to* 4.

THE SIMPLE RULES.

SIMPLE ADDITION.

37. A number that is exactly as large as two or more others taken together, is called the *sum* of those others.

Thus, 7 shillings and 3 shillings—taken together—are equal in amount to 10 shillings: for which reason the last number is said to be the "sum" of the other two.

38. ADDITION is an operation by which we find the sum of two or more numbers—that is, of course, numbers of the *same kind.*

Money is added to money, length to length, weight to weight, &c.; but when we say "What is the sum of 7 shillings and 3 days?" or "What is the sum of 8 yards of cloth and 5 pounds of tea?" we ask a question which the mind rejects as absurd.

39. In an exercise in Addition, the numbers whose sum is to be found are termed the *addends*.

40. Addition is called SIMPLE when the addends are simple numbers of the same denomination—that is, when the addends are *all* shillings, or *all* pence, or *all* miles, or *all* yards, or *all* apples, or *all* marbles, &c.

The addition of *abstract* numbers comes under the head of Simple Addition: because when, for instance, we say that the sum of 7 and 3 is 10, the addends (7 and 3) are necessarily presumed to be of the same denomination—although no denomination in particular may be contemplated. The sum of 7 *shillings* and 3 *shillings* is 10 *shillings*; of 7 *pence* and 3 *pence*, 10 *pence*; &c.: but the sum of 7 *shillings* and 3 *pence*, or of 7 *pence* and 3 *shillings*, is neither 10 *shillings* nor 10 *pence*.

Exercises in Addition, as in each of the other operations, divide themselves into two classes, which may be described as the *mental* class and the *slate* class.

Mental Exercises in Simple Addition.

Under this head we ask such questions as the following—beginning with concrete, and afterwards passing to abstract numbers:

How many marbles does a boy hold, who has 5 in one hand and 3 in the other? The pupils in a certain school are divided into three classes: there are 8 in the first class, 4 in the second, and 6 in the third; how many pupils in the school? A farmer bought 4 cows on Monday, 7 on Tuesday, 6 on Wednesday, and 5 on Thursday; how many did he buy altogether? What is the sum of 11, 8, and 9? Of 25, 7, 6. and 5? Of 36 and 18? [$36+18=36+8+10$: $36+8=44$; $44+10=54$.]

Slate Exercises in Simple Addition.

Under this head we begin with exercises like the following, of gradually increasing difficulty—there being, however, only *one* column of figures in each case:

					8
				6	9
			9	9	7
		4	7	8	6
	5	3	6	9	8
6	3	6	5	7	7
2	4	7	3	5	9
5	7	9	8	9	8
3	6	8	9	7	6
9	8	5	6	4	3
—	—	—	—	—	—

Applying himself to the first of these exercises (the one on the extreme left), the pupil—proceeding slowly at first—would say "9 and 3=12; 12 and 5=17; 17 and 2=19; 19 and 6=25." After a little while, however, the pupil should be required to proceed more quickly, mentioning none of the addends except the one with which he begins, and saying (in the case under consideration) "9—12—17—19—25;" or, if he began at the top of the column, "6—8—13—16—25."

From exercises such as these we pass to the addition of larger numbers—each expressed by two or more figures.

EXAMPLE I.—What is the sum of 365, 748, 856, 487, and 529?

These addends, being too large to be dealt with in their entirety, must be broken up—so to speak—into a number of parts; and the parts that naturally suggest themselves are those which the digits individually represent. We therefore break up the first addend into 3 groups of 100 each, 6 groups of 10 each, and 5 units; the second into 7 groups of 100 each, 4 groups of 10 each, and 8 units; the third into 8 groups of 100 each, 5 groups of 10 each, and 6 units; the fourth into 4 groups of 100 each, 8 groups of 10 each, and 7 units; and the fifth into 5 groups of 100 each, 2 groups of 10 each, and 9 units. When classified, or "sorted," the parts so obtained will stand thus:

Groups of 100 each.		Groups of 10 each.		Units.
3	+	6	+	5
7	+	4	+	8
8	+	5	+	6
4	+	8	+	7
5	+	2	+	9

Now, the value of the right-hand column of figures is (9+7+6+8+5=) 35 units; that is, 3 groups of 10 each, and 5 units. We therefore dispose of this column by setting down 5 units, and "carrying" 3 to the tens' column:

Groups of 100 each.		Groups of 10 each.		Units.
3	+	6		
7	+	4		
8	+	5		
4	+	8		
5	+	2		
		3	+	5

The value of the tens' column of figures—the carried figure, 3, being taken into account—is (3+2+8+5+4+6 =) 28 groups

of 10 each; in other words, 2 groups of 100 each, and 8 groups of 10 each. We therefore dispose of this column by setting down 8 groups of 10 each, and carrying 2 to the hundreds' column:

Groups of 100 each.		Groups of 10 each.		Units.
3				
7				
8				
4				
5				
2	+	8	+	5

The value of the hundreds' column of figures—the carried figure, 2, being taken into account—is (2+5+4+8+7+3=) 29 groups of 100 each; in other words, 2 groups of 1,000 each, and 9 groups of 100 each. The required *sum*, therefore, is—

Groups of 1,000 each.		Groups of 100 each.		Groups of 10 each.		Units.
2	+	9	+	8	+	5

or, when written more concisely,

2985.

This result is more easily obtained when we simply write the addends, one under the other, as shown in the margin—all the units' figures being in one vertical column, all the tens' figures in another, and all the hundreds' figures in another. The work then proceeds as before, except that the word "groups" is dropped, and that the "carried" figures, instead of being set down, are borne in mind*—the pupil saying "9—16—22—30—35; [set down] 5, and carry 3: [3]—5—13—18—22—28; [set down] 8, and carry 2: [2]—7—11—19—26—29.†

$$\begin{array}{r} 365 \\ 748 \\ 856 \\ 487 \\ 529 \\ \hline 2985 \end{array}$$

It will be seen that, of the three columns in the example before us, each of the others is dealt with in exactly the same way as the units' column. Thus, in the case of the tens' column, we say 3 (the carried figure) +2+8+5+4+6, instead of 30+20+80+50+40+60; and, in the case of the hundreds' column, 2 (the carried figure) +5+4+8+7+3, instead of 200+500+400+800+700+300.

41. Rule for Simple Addition (when there are no Decimals): Set down the addends—one under the

* This imposes no tax upon the memory, a carried figure being the very first that is dealt with in the next stage of the addition.

† Except in the case of beginners, the words enclosed in brackets are supposed to be omitted.

other—in such a way that all the figures occupying the same place shall stand in the same vertical column. Then, having drawn a horizontal line to separate the addends from the sum, find what number the digits in the units' column, taken together, represent. Write the units' figure of this number at the bottom of the units' column, and carry the tens' figure—if there be one—to the next column. Treat the tens' and every other column in the same way—regarding each, in turn, as the units' column, and always taking the carried figure, when there is one, into account. Should there be a figure to carry from the last column, (or from the highest place occupied by any one of the digits of the addends,) set this figure down as the most left-hand one of the required sum.

NOTE 1.—If the addends, instead of being arranged as directed by the rule, were set down in some such careless way as that shown in the margin, the pupil would not be unlikely to make a wrong classification of the digits—mistaking, for instance, a tens' figure for a units' or a hundreds' figure.

$$\begin{array}{l} \;\;365 \\ \;\;\;\;748 \\ 856 \\ \;\;\;487 \\ \;529 \\ \hline \end{array}$$

NOTE 2.—By beginning with the units' column, we avoid the inconvenience of being obliged—as we should be, in most cases, if we began with the most left-hand column—to alter one or more figures. Thus, returning to Example I., and proceeding from left to right, we first set down 7 at the bottom of the hundreds' column (5+4+8+7+3=27), and 2 in the next place on the left. We next write 5 at the bottom of the tens' column (2+8+5+4+6=25), and 2 in the next place on the left—already occupied by 7, which, therefore, must be changed into (7+2=) 9. And we finish by writing 5 at the bottom of the units' column (9+7+6+8+5=35) and 3 in the next place on the left—already occupied by 5, which, therefore, must be changed into (5+3=) 8.

$$\begin{array}{r} 365 \\ 748 \\ 856 \\ 487 \\ 529 \\ \hline 2\cancel{7}\cancel{5}5 \\ 98\;\; \end{array}$$

[ADDITION OF DECIMALS.]

EXAMPLE II.—Find ·365+·748+·856+·487+·529.

For a reason already explained, we begin by arranging the addends in the way shown in the margin, so that all the figures

occupying the same place shall stand in the same vertical column. We then see that the value of the right-hand column of figures is $(9+7+6+8+5=)$ 35 thousandths: we therefore write 5 at the bottom of the column, and carry 3 to the middle column. We next see that—the carried figure, 3, being taken into account—the value of the middle column is $(3+2+8+5+4+6=)$ 28 hundredths: we therefore write 8 at the bottom of the column, and carry 2 to the left-hand column. Lastly, we see that—the carried figure, 2, being taken into account—the value of the left-hand column is $(2+5+4+8+7+3=)$ 29 tenths: we therefore write 9 at the bottom of the column, and 2 in the next place on the left—that is, in the units' place. Writing the decimal point to mark the units' figure (2), we thus find the required sum to be 2·985.

$$\begin{array}{r} \cdot365 \\ \cdot748 \\ \cdot856 \\ \cdot487 \\ \cdot529 \\ \hline 2\cdot985 \end{array}$$

EXAMPLE III.—Find $3\cdot65+7\cdot48+8\cdot56+4\cdot87+5\cdot29$.

Having arranged the addends in the way shown in the margin, we dispose of the right-hand column of figures, the value of which is $(9+7+6+8+5=)$ 35 hundredths, by writing 5 at the bottom of the column, and carrying 3 to the middle column. We next dispose of the middle column, the value of which—the carried figure, 3, being taken into account—is $(3+2+8+5+4+6=)$ 28 tenths, by writing 8 at the bottom of the column, and carrying 2 to the left-hand column. And we dispose of the left-hand column, the value of which—the carried figure, 2, being taken into account—is $(2+5+4+8+7+3=)$ 29 units, by writing 9 at the bottom of the column—that is, in the units' place,—and 2 in the next place on the left. The decimal point being written to mark the units' figure (9), the required sum is thus found to be 29·85.

$$\begin{array}{r} 3\cdot65 \\ 7\cdot48 \\ 8\cdot56 \\ 4\cdot87 \\ 5\cdot29 \\ \hline 29\cdot85 \end{array}$$

Comparing the last two examples with the first, we see that, once the addends are properly arranged, the addition of decimals proceeds in exactly the same way as the addition of whole numbers—the digits being invariably called by their respective names, and each column of figures, in turn, being regarded as the units' column. Because, in practice we never speak of "tenths," or "hundredths," or "thousandths," &c., such expressions being only employed for the sake of explanation. In all three examples, the value of the right-hand column of figures, for instance, is expressed as "*thirty-five*"—a word which at once suggests the digits 3 and 5; and without stopping to ascertain *what* place the figures of the column occupy, we are able, from our knowledge of notation, to say that 5 must be written in the *same* place, and 3 in the next place on the left—5 representing *five* of the thirty-five, and 3 the remaining *thirty*.

42. Rule for Simple Addition, when there are Decimals: Arrange the addends so as to have all the decimal points in a vertical line (that all the figures occupying the same place may stand in the same vertical column); then, proceed as if there were no Decimals, and write the decimal point in the result in the same vertical line as the decimal points of the addends.

NOTE.—When the decimals do not all occupy an equal number of places, we usually equalise the number of places by annexing one or more ciphers. Thus, in finding the sum of 23·45, 56·789, and 87·6, we usually write 23·45 as 23·450, and 87·6 as 87·600.

23·450
56·789
87·600
167·839

43. PROVING an operation means—employing some test which satisfies us that the operation has been correctly performed. To "prove" an exercise in Addition—whether it be Simple, Compound, or Fractional Addition—we go over the work a second time, beginning at the top, and adding downwards, (having, the first time, begun at the bottom, and added upwards,)—that the digits may not occur in the same order as before. When the same result is obtained in both cases, we conclude that no mistake has been committed.

The pupil who, in proceeding upwards, falls into the mistake of saying, for example, $17+8=23$, would not be at all unlikely to repeat this mistake if the digits were taken in the same order the second time; and although it is quite possible that, when the order is reversed, the question $17+8=?$ will again present itself, yet such coincidences are very exceptional in practice.

SIMPLE SUBTRACTION.

44. By the *difference* between two numbers is meant—what would remain of the larger, if a portion equal to the smaller number were taken away.

Thus, the "difference" between 10 shillings and 7 shillings is 3 shillings: because if, out of a purse containing 10 shillings, 7 shillings were taken, 3 shillings would be left.

45. SUBTRACTION is an operation by which we find the difference between two numbers, which, however, must be of the *same kind.*

Money is subtracted from money, length from length, weight from weight, &c.; but we never, except in jest, speak of the difference between (say) 9 miles and 4 years, or between 12 gallons of milk and 8 acres of land.

46. In an exercise in Subtraction, the larger of the two numbers whose difference we want to find is called the *minuend;* the smaller of the two is termed the *subtrahend.* The difference, when found, is commonly spoken of as the *remainder.*

Thus, in the exercise "What is the difference between 10 shillings and 7 shillings?" the "minuend" is 10 shillings; the "subtrahend," 7 shillings; and the "difference," or "remainder," 3 shillings.

47. Subtraction is called SIMPLE when the minuend and the subtrahend are simple numbers of the same denomination.

The subtraction of an *abstract* number from another comes under the head of Simple Subtraction: because it would be absurd to say, for example, that the difference between 10 and 7 is 3, if the numbers 10 and 7 were not presumed to be of the same denomination. The difference between 10 *shillings* and 7 *shillings* is 3 *shillings;* between 10 *pence* and 7 *pence*, 3 *pence;* &c.: but the difference between 10 *shillings* and 7 *pence*, or between 10 *pence* and 7 *shillings*, is neither 3 *shillings* nor 3 *pence.*

MENTAL EXERCISES IN SIMPLE SUBTRACTION.

Under this head would come exercises like the following—concrete numbers being dealt with first, and abstract numbers afterwards:

A boy bought 10 marbles, and lost 4 of them; how many had he left? A girl was sent to market with a dozen (12) eggs, and on the way she broke 5 of them; how many remained unbroken? There are 17 children in a school: how many would remain if 8 were sent home?—What is the difference between 15 and 23? [Difference between 15 and 20 = 5; between 15 and 23=5+3=8.] Between 26 and 35? [Difference between 26 and 30=4; between 26 and 35=4+5=9.] Between 37 and 50? [Difference between 37 and 40=3; between 37 and 50=3+10=13.]

At the proper time, the pupil should be required to complete, and to commit to memory, the following table, which embraces every question that can arise at any particular stage of a slate exercise in Simple Subtraction. The first and last columns involve scarcely any calculation:—

SUBTRACTION TABLE.

1 from	2 from	3 from	4 from	5 from
1 = 0	2 = 0	3 = 0	4 = 0	5 = 0
2 " 1	3 " 1	4 " 1	5 " 1	6 " 1
3 " 2	4 " 2	5 " 2	6 " 2	7 " 2
4 " 3	5 " 3	6 " 3	7 " 3	8 " 3
5 " 4	6 " 4	7 " 4	8 " 4	9 " 4
6 " 5	7 " 5	8 " 5	9 " 5	10 " 5
7 " 6	8 " 6	9 " 6	10 " 6	11 " 6
8 " 7	9 " 7	10 " 7	11 " 7	12 " 7
9 " 8	10 " 8	11 " 8	12 " 8	13 " 8
10 " 9	11 " 9	12 " 9	13 " 9	14 " 9
6 from	**7 from**	**8 from**	**9 from**	**10 from**
6 = 0	7 = 0	8 = 0	9 = 0	10 = 0
7 " 1	8 " 1	9 " 1	10 " 1	11 " 1
8 " 2	9 " 2	10 " 2	11 " 2	12 " 2
9 " 3	10 " 3	11 " 3	12 " 3	13 " 3
10 " 4	11 " 4	12 " 4	13 " 4	14 " 4
11 " 5	12 " 5	13 " 5	14 " 5	15 " 5
12 " 6	13 " 6	14 " 6	15 " 6	16 " 6
13 " 7	14 " 7	15 " 7	16 " 7	17 " 7
14 " 8	15 " 8	16 " 8	17 " 8	18 " 8
15 " 9	16 " 9	17 " 9	18 " 9	19 " 9

SLATE EXERCISES IN SIMPLE SUBTRACTION.

EXAMPLE I.—What is the difference between 798 and 325?

These numbers, on account of their largeness, must each be broken up into the parts which the digits individually represent—the minuend into 7 groups of 100 each, 9 groups of 10 each, and 8 units; and the subtrahend into 3 groups of 100 each, 2 groups of 10 each, and 5 units. We then subtract the parts of the subtrahend, one by one. When 3 hundreds are taken from 7 hundreds, 4 hundreds remain; when 2 tens are taken from 9 tens, 7 tens remain; and when 5 units are taken from 8 units, 3 units remain. So that the required difference is 473.

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798
325
---
473
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NOTE.—In this case it is immaterial whether we begin at the right or at the left, there being no "carrying."

EXAMPLE II.—What is the difference between 83,705 and 24,568?

Here an apparent difficulty is presented by the fact that some of the figures of the subtrahend are greater in value than the corresponding figures of the minuend. Let us suppose the minuend to be a sum of money, consisting of 8 *ten-thousand-pound* notes, 3 *one-thousand-pound* notes, 7 *one-hundred-pound* notes, and 5 *one-pound* notes; and that, with this amount in our possession, we want to pay 24,568 pounds, in five instalments of 20,000 pounds, 4,000 pounds, 500 pounds, 60 pounds, and 8 pounds—represented by the digits 2, 4, 5, 6, and 8, respectively.

With 2 of the 8 *ten-thousand-pound* notes we pay the first instalment (20,000 pounds). This reduces the number of *ten-thousand-pound* notes to (8—2=) 6. To pay the second instalment (4,000 pounds), we require 4 *one-thousand-pound* notes, but we have only 3. We therefore take one of the 6 *ten-thousand-pound* notes, and change it into 10 *one-thousand-pound* notes. We then have—of *ten-thousand-pound* notes, (6—1=) 5; and of *one-thousand-pound* notes, (10+3=) 13. Paying the second instalment with 4 of the 13 *one-thousand-pound* notes, we have (13—4=) 9 such notes remaining. With 5 of the 7 *one-hundred-pound* notes we pay the third instalment (500 pounds). This reduces the number of *one-hundred-pound* notes to (7—5=) 2. To pay the fourth instalment (60 pounds), we require 6 *ten-pound* notes. Having no such notes, we take one of the 2 *one-hundred-pound* notes, and change it into 10 *ten-pound* notes. This reduces the number of *one-hundred* pound notes to (2—1=) 1. Paying the fourth instalment with 6 of the 10 *ten-pound* notes, we have (10—6=) 4 such notes remaining. To pay the last instalment (8 pounds), we require 8 *one-pound* notes, but we have only 5. We therefore take one of the 4 *ten-pound* notes, and change it into 10 *one-pound* notes. We then have—of *ten-pound* notes, (4—1=) 3; and of *one-pound* notes, (10+5=) 15. Paying the last instalment with 8 of the 15 *one-pound* notes, we have (15—8=) 7 such notes remaining. So that, after payment of all the instalments, there remain—of *ten-thousand-pound* notes, 5; of *one-thousand-pound* notes, 9; of *one-hundred-pound* notes, 1; of *ten-pound* notes, 3; and of *one-pound* notes, 7: that is, 59,137 pounds.

$$\begin{array}{r} 83,705 \\ \underline{24,568} \\ \cancel{6}9,\cancel{2}\cancel{4}7 \\ 5\;\;\;13\;\; \end{array}$$

It will be seen that, in proceeding from left to right in the manner just explained, we set down certain figures which we are afterwards obliged to alter: for 6, 2, and 4 we have to substitute 5, 1, and 3, respectively. We shall avoid this inconvenience by considering, beforehand, what "change" is necessary, and supposing it to have been all obtained previously to the payment of any of the instalments. Passing from left to right, and comparing the figures of the subtrahend *with the corresponding* ones of the minuend, we see at a glance

that one of the 8 *ten-thousand-pound* notes must be changed into *one-thousand-pound* notes, of which there will then be (10+3=) 13; that one of the 7 *one-hundred-pound* notes must be changed into *ten-pound* notes, of which there will then be (10+0=) 10; and that one of the 10 *ten-pound* notes must be changed into *one-pound* notes, of which there will then be (10+5=) 15.

1 11
83,705
24,568
59,137

Let us suppose the three notes in question to have been changed, and let us write 1 over 8 to represent the first note, 1 over 7 to represent the second, and 1 over 0 to represent the third. We then see that, after payment of the first instalment, the 8 *ten-thousand-pound* notes will have been diminished—not merely by 2, but—by 1 (the changed note)+2; that, after payment of the second instalment, the 13 *one-thousand-pound* notes will have been diminished by 4; that, after payment of the third instalment, the 7 *one-hundred-pound* notes will have been diminished—not merely by 5, but—by 1 (the changed note)+5; that, after payment of the fourth instalment, the 10 *ten-pound* notes will have been diminished—not merely by 6, but—by 1 (the changed note)+6; and that, after payment of the last instalment, the 15 *one-pound* notes will have been diminished by 8. We therefore obtain the most left-hand figure (5) of the remainder by subtracting 1+2 from 8; the next figure (9), by subtracting 4 from 13; the next figure (1), by subtracting 1+5 from 7; the next figure (3), by subtracting 1+6 from 10; and the most right-hand figure (7), by subtracting 8 from 15.

By proceeding from right to left, instead of from left to right, we are able to dispense with the setting down of *ones* to represent the notes that we suppose to have been changed. Thus, seeing that the digit 8 is greater in value than 5, we say—[take] not 8 from 5, but—8 from 15, and 7 [will remain.] Then, by the mere fact of our having called 5 *fifteen* we are reminded of the 1 to be added—or "carried," as it is called—to the next figure (6) of the subtrahend. We therefore say 1 and 6=7; [take 7] from 10, and 3 [will remain.] By the fact of our having called 0 *ten* we are reminded of the 1 to be carried to the next figure (5) of the subtrahend. We therefore say 1 and 5=6; [take 6] from 7, and 1 [will remain.] Not having called 7 seven*teen* we know that 1 is *not* to be carried to the next figure (4) of the subtrahend. We therefore say—[take] not 4 from 3, but—4 from 13, and 9 [will remain.] The fact of our having called 3 *thirteen* reminds us of the 1 to be carried to the most left-hand figure (2) of the subtrahend. We therefore say 1 and 2=3; [take 3] from 8, and 5 [will remain.]*

* Except in the case of beginners, the words in brackets are supposed to be omitted.

48. Rule for Simple Subtraction (when there are no Decimals): Write the subtrahend under the minuend, in such a way that every two figures belonging to the same place shall stand in the same vertical column. Draw a horizontal line to separate the subtrahend from the remainder. Then, if the units' figure of the subtrahend be not greater in value than that of the minuend, take the value of the former figure from that of the latter, and set down the result in the units' place of the remainder. But if the units' figure of the subtrahend be greater in value than that of the minuend, take the value of the former figure from that of the latter *increased by* 10 (0 in the minuend being read "*ten;*" 1, "*eleven;*" 2, "*twelve;*" 3, "*thirteen;*" &c.—as the case may be); write the result in the units' place of the remainder; and carry 1 to the next figure of the subtrahend. Treat the tens' and every other column in the same way—regarding each, in turn, as the units' column, and taking the carried 1 into account in every case in which there *is* 1 to carry.

NOTE.—When the minuend occupies a greater number of places than the subtrahend, we proceed as if the two sets of figures had been made equal in number by the prefixing of one or more ciphers to the subtrahend. Thus, in finding the difference between 13,568 and 729, we proceed as if the subtrahend, 729, were written under the form 00729.

$$\begin{array}{r} 13568 \\ 729 \\ \hline 12839 \end{array}$$

[SUBTRACTION OF DECIMALS.]

EXAMPLE II.—What is the difference between 83·705 and 24·568?

Here we have—in the minuend, 8 *tens*, 3 *units*, 7 *tenths*, (no *hundredths*,) and 5 *thousandths;* and in the subtrahend—2 *tens*, 4 *units*, 5 *tenths*, 6 *hundredths*, and 8 *thousandths*. Now, if, before proceeding with the subtraction, we consider what changes are necessary in the minuend, we shall find, as we pass from left to right, that one of the 8 *tens* must be converted into *units*, of which there will then be *(10+3=) 13*; that one of the 7 *tenths* must be converted *into hundredths, of* which there will then be (10+0=) *10; and that one of the* 10 *hundredths* must be converted into

$$\begin{array}{r} \scriptstyle 1\ \ \ 11 \\ 83{\cdot}705 \\ 24{\cdot}568 \\ \hline 59{\cdot}137 \end{array}$$

thousandths, of which there will then be (10+5=) 15. Noting these changes, by writing the digit 1 over 8, over 7, and over 0, we see that, after the substraction, the 15 *thousandths* will have been diminished by 8; the 10 *hundredths* by the sum of 1 and 6; the 7 *tenths* by the sum of 1 and 5; the 13 *units* by 4; and the 8 *tens* by the sum of 1 and 2. The remainder (written from right to left) is thus found to consist of 7 *thousandths*, 3 *hundredths*, 1 *tenth*, 9 *units*, and 5 *tens*. In finding the figures of the remainder, we omit the words "thousandths," "hundredths," "tenths," &c.,—saying, simply, "8 from 15 and 7; [carry] 1 to 6—7; [7] from 10, and 3; [carry] 1 to 5—6; [6] from 7, and 1; &c. So that the work is exactly the same as in Ex. I., except that the decimal point must be written after the units' figure, 9.

EXAMPLE III.—What is the difference between 837·05 and 245·68?

Here we have—in the minuend, 8 *hundreds*, 3 *tens*, 7 *units*, (no *tenths*,) and 5 *hundredths*; and in the subtrahend—2 *hundreds*, 4 *tens*, 5 *units*, 6 *tenths*, and 8 *hundredths*. When, in this case, we consider the changes which are necessary in the minuend, we see that one of the 8 *hundreds* must be converted into *tens*, of which there will then be (10+3=) 13; that one of the 7 *units* must be converted into *tenths*, of which there will then be (10+0=) 10; and that one of the 10 *tenths* must be converted into *hundredths*, of which there will then be (10+5=) 15. After the subtraction, therefore, the 15 *hundredths* will have been diminished by 8; the 10 *tenths* by the sum of 1 and 6; the 7 *units* by the sum of 1 and 5; the 13 *tens* by 4; and the 8 *hundreds* by the sum of 1 and 2. So that in the remainder we shall have—7 *hundredths*, 3 *tenths*, 1 *unit*, 9 *tens*, and 5 *hundreds*. It will be seen that in this case, also, we proceed as in Ex. I.,—taking care, of course, to write the decimal point after the units' figure, 1.

```
1 1 1
837·05
245·68
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591·37
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49. Rule for Simple Subtraction, when there are Decimals: Place the subtrahend under the minuend, in such a way that the two decimal points shall stand in the same vertical line; then, proceed as if there were no Decimals, and write the decimal point in the remainder directly under the other two decimal points.

NOTE.—When the decimal in the minuend and that in the subtrahend do not occupy an equal number of places, we *equalise the number of places* by annexing one or more ciphers.

Thus, in subtracting 4·56 from 7·3 we write 7·3 as 7·30; in subtracting ·8 from 12·345 we write ·8 as ·800; in subtracting ·789 from 654 we write 654 as 654·000; &c. :—

7·30	12·345	654·000
4·56	·800	·789
2·74	11·545	653·211

50. To *prove* an exercise in Subtraction—whether it be Simple, Compound, or Fractional Subtraction—we add the remainder to the subtrahend; and when the minuend is obtained for result, the work is presumed to have been correctly performed.

This is easily understood. A person who spends a portion of his money, and afterwards gets back this portion, or an equal amount, finds himself in possession of the original sum. The portion spent is the "subtrahend;" the unspent portion, the "remainder;" and the original sum, the "minuend."

NOTE.—In explaining the principle upon which, in an exercise in Subtraction, 1 is occasionally "carried," the teacher should carefully avoid the use of the word *borrow*, which occurs in too many treatises on Arithmetic. We constantly hear a child or a class of children told to "borrow" 1 from a figure in the minuend, (this figure being sometimes 0, which has nothing to lend,) and to "pay it back"—not to the figure which is supposed to have lent it, but—to a figure in the subtrahend. Can anything be more absurd than this? We are able to understand the case of a person who borrows money when he wants to pay a larger sum than he has at his command; but in an exercise in Subtraction the minuend is never less than the subtrahend—so that there is no *necessity* for borrowing. As well might a man be said to "borrow" when, on buying a 15-shilling hat, and finding an insufficient amount of silver in his purse, he is obliged to change a sovereign.

SIMPLE MULTIPLICATION.

51. MULTIPLICATION is an operation by which, without actually performing the addition, we find what a number would amount to if added to itself a given number of times.

52. In an exercise in Multiplication, the number to be repeated as addend—or, as we say, to be "multiplied"—is called the *multiplicand;* the

number indicating how many times the multiplicand is to be taken as addend is called the *multiplier;* the result obtained is termed the *product.* The multiplicand and the multiplier are often spoken of, collectively, as the *factors* (*i.e.*, the makers or producers) of the product.

We can find the number of pence in 9 shillings, for instance, by setting down 12, the number of pence in one shilling, 9 different times, and then adding all the *twelves* together; but, knowing the Multiplication Table, we are able to proceed more expeditiously—setting down 12 once only, writing 9 (the *number* of twelves) under it, and then saying "9 *twelves*=108." Here 108 is the "product" of the "factors" 12 and 9, of which factors 12 is the "multiplicand" and 9 the "multiplier"—12 having, as we technically express it, been *multiplied by* 9.

$$\begin{array}{r} 12 \\ 12 \\ 12 \\ 12 \\ 12 \\ 12 \\ 12 \\ 12 \\ 12 \\ \hline 108 \end{array} \qquad \begin{array}{r} 12 \\ 9 \\ \hline 108 \end{array}$$

It is hardly necessary to observe that the sum of 9, 8, 6, 5, and 3, for example, could *not* be found by Multiplication,—the addends not being *all the same.*

$$\begin{array}{r} 9 \\ 8 \\ 6 \\ 5 \\ 3 \\ \hline 31 \end{array}$$

In finding the product of two numbers, we are at liberty to take either of the two for multiplicand, and the other for multiplier. Thus, the asterisks in the margin form 7 rows of 4 each, for which reason we can ascertain the entire number by taking 4 as addend 7 times—that is, multiplying 4 by 7; the asterisks may also be regarded as forming 4 rows of 7 each, for which reason we can ascertain the entire number by taking 7 as addend 4 times—that is, multiplying 7 by 4. So that 7 *fours* are the same as 4 *sevens.* It could easily be shown, in the same way, that 8 *fives* are the same as 5 *eights*; that 13 *sixes* are the same as 6 *thirteens;* &c. In practice, however, we usually* take the larger factor for multiplicand, and the smaller for multiplier—it being much easier to multiply 365 by 9, for instance, than to multiply 9 by 365.

```
   7 4 | 28
  7  7  7  7
  *  *  *  * ...4
  *  *  *  * ...4        4
  *  *  *  * ...4        7
  *  *  *  * ...4       ---
  *  *  *  * ...4       28
  *  *  *  * ...4
  *  *  *  * ...4
```

* Except when the multiplier happens to be a power of 10.

53. The multiplier is always an *abstract* number.

In finding the number of pence in 9 shillings, we multiply 12 pence (not by 9 *shillings*, but) by the abstract number 9: in other words, we find what 12 pence would amount to if taken 9 times—not 9 *shillings* times—as addend.

54. Multiplication is called SIMPLE when the factors are simple numbers.

55. In Simple Multiplication, the product is always of the same denomination as the multiplicand.

Thus, the double of 5 *pence* is 10 *pence;* of 5 *shillings*, 10 *shillings;* of 5 *yards*, 10 *yards;* of 5 *days*, 10 *days;* &c. Even when, without mentioning any denomination in particular, we speak of 10 as the double of 5, the numbers 10 and 5 are necessarily presumed to be of the same denomination. The expression "2 *fives*=10" is never either intended or understood to mean that the double of 5 *pence* is 10 *shillings*, or that the double of 5 *yards* is 10 *days*.

MULTIPLICATION TABLE.

Two			Three			Four			Five		
ones	=	2	*ones*	=	3	*ones*	=	4	*ones*	=	5
twos	,,	4	*twos*	,,	6	*twos*	,,	8	*twos*	,,	10
threes	,,	6	*threes*	,,	9	*threes*	,,	12	*threes*	,,	15
fours	,,	8	*fours*	,,	12	*fours*	,,	16	*fours*	,,	20
fives	,,	10	*fives*	,,	15	*fives*	,,	20	*fives*	,,	25
sixes	,,	12	*sixes*	,,	18	*sixes*	,,	24	*sixes*	,,	30
sevens	,,	14	*sevens*	,,	21	*sevens*	,,	28	*sevens*	,,	35
eights	,,	16	*eights*	,,	24	*eights*	,,	32	*eights*	,,	40
nines	,,	18	*nines*	,,	27	*nines*	,,	36	*nines*	,,	45
tens	,,	20	*tens*	,,	30	*tens*	,,	40	*tens*	,,	50
elevens	,,	22	*elevens*	,,	33	*elevens*	,,	44	*elevens*	,,	55
twelves	,,	24	*twelves*	,,	36	*twelves*	,,	48	*twelves*	,,	60

Six			Seven			Eight			Nine		
ones	=	6	*ones*	=	7	*ones*	=	8	*ones*	=	9
twos	,,	12	*twos*	,,	14	*twos*	,,	16	*twos*	,,	18
threes	,,	18	*threes*	,,	21	*threes*	,,	24	*threes*	,,	27
fours	,,	24	*fours*	,,	28	*fours*	,,	32	*fours*	,,	36
fives	,,	30	*fives*	,,	35	*fives*	,,	40	*fives*	,,	45
sixes	,,	36	*sixes*	,,	42	*sixes*	,,	48	*sixes*	,,	54
sevens	,,	42	*sevens*	,,	49	*sevens*	,,	56	*sevens*	,,	63
eights	,,	48	*eights*	,,	56	*eights*	,,	64	*eights*	,,	72
nines	,,	54	*nines*	,,	63	*nines*	,,	72	*nines*	,,	81
tens	,,	60	*tens*	,,	70	*tens*	,,	80	*tens*	,,	90
elevens	,,	66	*elevens*	,,	77	*elevens*	,,	88	*elevens*	,,	99
twelves	,,	72	*twelves*	,,	84	*twelves*	,,	96	*twelves*	,,	108

Ten			Eleven			Twelve		
ones	=	10	*ones*	=	11	*ones*	=	12
twos	„	20	*twos*	„	22	*twos*	„	24
threes	„	30	*threes*	„	33	*threes*	„	36
fours	„	40	*fours*	„	44	*fours*	„	48
fives	„	50	*fives*	„	55	*fives*	„	60
sixes	„	60	*sixes*	„	66	*sixes*	„	72
sevens	„	70	*sevens*	„	77	*sevens*	„	84
eights	„	80	*eights*	„	88	*eights*	„	96
nines	„	90	*nines*	„	99	*nines*	„	108
tens	„	100	*tens*	„	110	*tens*	„	120
elevens	„	110	*elevens*	„	121	*elevens*	„	132
twelves	„	120	*twelves*	„	132	*twelves*	„	144

ıe invention of this table is attributed to Pythagoras, a ırated Greek philosopher, who was born about the year B.C. Every schoolboy should be required to construct able for himself—the teacher affording the necessary assist- by indicating, in some such way as that shown below, the ions to be performed:

1	2	3	4	
1	2	3	4	
——	——	——	——	
ıes=)	; (2 *twos*=)	; (2 *threes*=)	; (2 *fours*=)	; &c.
1	2	3	4	
1	2	3	4	
1	2	3	4	
——	——	——	——	
ıes=)	; (3 *twos*=)	; (3 *threes*=)	; (3 *fours*=)	; &c.
1	2	3	4	
1	2	3	4	
1	2	3	4	
1	2	3	4	
——	——	——	——	
ıes=)	; (4 *twos*=)	; (4 *threes*=)	; (4 *fours*=)	; &c.
&c.	&c.	&c.	&c.	

Mental Exercises in Simple Multiplication.

ıder this head are asked questions like the following—all em particular applications of the Multiplication Table: 'farm is divided into 3 fields, in each of which there are 8); how many sheep on the farm? A person travels at the of 5 miles an hour; how far does he go in 9 hours? Milk ; sold at 10 pence a gallon, what is the price of 12 gallons?

Slate Exercises in Simple Multiplication.

dealing with the exercises which come under this head, aturally proceed in the following order:—

CASE I., in which the multiplicand exceeds 12, and the multiplier is either a power of 10 or a number not greater than 12.

EXAMPLE I.—What is the product of 47 and 1,000?

Setting down 47, we perform the multiplication by simply removing the decimal point (supposed to stand after the units' figure, 7) three places to the right—that is, as many places as there are ciphers in 1,000 : $47 \times 1{,}000 = 47{,}000$.* [See § 21.]

EXAMPLE II.—What is the product of 468 and 7?

Here we regard the multiplicand as broken up into the parts which its digits individually represent—namely, 4 groups of 100 each, 6 groups of 10 each, and 8 units; and we then deal with those parts, one by one. Multiplying the 8 units by 7, we obtain 56 units, which we express by writing 6 in the units' place of the product, and carrying 5 to the tens' place. Next, multiplying the 6 tens by 7, we obtain 42 tens: adding 5, the number of tens carried, we have, altogether, $(42+5=)$ 47 tens; and this number we express by writing 7 in the tens' place of the product, and carrying 4 to the hundreds' place. Lastly, multiplying the 4 hundreds by 7, we obtain 28 hundreds, to which we add 4, the number of hundreds carried; we then have, altogether, $(28+4=)$ 32 hundreds, which we express by writing 2 in the hundreds' place of the product, and 3 in the next place on the left. So that the required product is 3,276. A person unacquainted with Multiplication would be obliged, in obtaining this result, to proceed in the way shown in the margin—setting down 468, as addend, 7 different times, and then adding in the ordinary way.

$$\begin{array}{r} 468 \\ 7 \\ \hline 3276 \end{array}$$

$$\begin{array}{r} 468 \\ 468 \\ 468 \\ 468 \\ 468 \\ 468 \\ 468 \\ \hline 3276 \end{array}$$

NOTE.—If, in multiplying 468 by 7, we began at the left-hand side, we should have to set down, successively, $(4 \times 7=)$ 28 hundreds, $(6 \times 7=)$ 42 tens, and $(8 \times 7=)$ 56 units. The setting down of the 42 tens would involve the "carrying" of 4 to the hundreds' place of the product—in other words, the substitution of 2 hundreds for 8 hundreds, and of 3 thousands for 2 thousands; whilst the setting down of the 56 units would involve the carrying of 5 to the tens' place of the product—in other words, the substitution of 7 tens for 2 tens.

$$\begin{array}{r} 468 \\ \underline{7} \\ \cancel{2}8\cancel{2}6 \\ 327 \end{array}$$

* Such questions as this might be treated as *mental* exercises.

56. Rule for Simple Multiplication, (1.) when the multiplier is either 10 or a power of 10: Remove the decimal point of the multiplicand as many places to the right as there are ciphers in the multiplier. (2.) When the multiplier is not a power of 10, and does not exceed 12: Multiply the units' figure of the multiplicand by the multiplier; write the units' figure of the result in the units' place of the required product, and carry the tens' figure, if there be one, to the tens' place of the product. Treat the tens' and every other figure of the multiplicand in the same way—regarding each, in turn, as the units' figure, and always taking the carried figure, when there is one, into account.

CASE II., in which both the multiplicand and the multiplier exceed 12, but the multiplier is resolvable into two factors—each of them either a power of 10 or a number not greater than 12.

EXAMPLE III.—An orchard contains 24 apple-trees, on each of which there are 567 apples; how many apples in the orchard?

Here we have to find a number 24 times as large as 567—that is, to multiply 567 by 24. As factors of the multiplier we can take 6 and 4, or 8 and 3, or 12 and 2. This being the case, we can regard the apple-trees as forming 4 rows of 6 each, or 3 rows of 8 each, or 2 rows of 12 each. If the trees were in rows of 6 each, the number of apples on one row would be 567 ×6=3,402; and as there would be 4 such rows altogether, the number of apples in the orchard must be 3,402×4=13,608. If the trees were in rows of 8 each, the number of apples on one row would be 567×8=4,536; and as there would be 3 such rows altogether, the number of apples in the orchard must be 4,536×3=13,608—the result already obtained. If the trees were in rows of 12 each, the number of apples on one row would be 567×12=6,804; and as there would be 2 such rows altogether, the number of apples in the orchard must be—as before—6,804×2=13,608. So that we obtain 24 times a number when we multiply 6 times the number by 4, or 8 times the number by 3, or 12 times the number by 2.

567	567	567
6	8	12
3402	4536	6804
4	3	2
13608	13608	13608

EXAMPLE IV.—What is the product of 456 by 800?

Here the multiplier is resolvable into the factors 8 and 100. Upon the principle just explained, therefore, we obtain the required product by multiplying 456 by 8, and the result by 100:—$456 \times 8 = 3{,}648$; $3{,}648 \times 100 = 364{,}800$.

57. **Rule for Simple Multiplication, when the multiplier exceeds 12, and is not a power of 10, but is resolvable into two factors—each of them either a power of 10 or a number not greater than 12: Multiply the multiplicand by one of the factors, and the resulting product by the other factor.**

CASE III., in which the multiplier is greater than 12, and is neither a power of 10 nor a number resolvable into factors.

EXAMPLE V.—Multiply 6,789 by 234.

Here we break up the multiplier into the parts which its digits individually represent—namely, 4, 30, and 200; and we then deal with those parts, one by one. A knowledge of Case I. enables us to find 4 times the multiplicand:—$6{,}789 \times 4 = 27{,}156$. A knowledge of Case II. enables us to find 30 times the multiplicand, 30 being resolvable into the factors 3 and 10:—$6{,}789 \times 3 = 20{,}367$; $20{,}367 \times 10 = 203{,}670$. A knowledge of Case II. also enables us to find 200 times the multiplicand, 200 being resolvable into the factors 2 and 100:—$6{,}789 \times 2 = 13{,}578$; $13{,}578 \times 100 = 1{,}357{,}800$. Adding together these "partial" products, as they are called,—that is, 4 times, 30 times, and 200 times the multiplicand,—we thus find 234 times the multiplicand to be 1,588,626:*

$$
\begin{array}{rl}
6789 & \\
234 & \\
\hline
27156 = & 4 \text{ times } 6{,}789 \\
203670 = & 30 \quad ,, \quad ,, \\
1357800 = & 200 \quad ,, \quad ,, \\
\hline
1588626 = & 234 \quad ,, \quad ,,
\end{array}
$$

In practice, we dispense with the cipher at the end of the second,

* Just as, if there were 234 boxes, each containing 6,789 oranges, we should find the total number of oranges ($6{,}789 \times 234$) by adding together the contents of 4 boxes ($6{,}789 \times 4$), the contents of 30 boxes ($6{,}789 \times 30$), and the contents of 200 boxes ($6{,}789 \times 200$).

as well as with the two ciphers at the end of the third partial product: it being sufficient to multiply by 4, 3, and 2, successively, and to set down the resulting products—one below the other—in such a way that the most right hand-figure (6) of the first shall stand in the units' place; the most right-hand figure (7) of the second, in the tens' place; and the most right-hand figure (8) of the third, in the hundreds' place—these being the places occupied respectively by 4, 3, and 2, the digits of the multiplier.

6789
234
27156
20367
13578
1588626

EXAMPLE VI.—Multiply 342,819 by 50,607.

We first multiply by 7, and set down the most right-hand figure (3) of the result under 7. We next multiply by 6, and set down the most right-hand figure (4) of the result under 6. We then multiply by 5, and set down the most right-hand figure (5) of the result under 5. Adding together the three results thus obtained, we find the required product to be 17,349,041,133:—

342819			
50607			
2399733	= 7	times	342,819
2056914	= 600	„	„
1714095	= 50,000	„	„
17349041133	= 50,607	„	„

Here, what appears to be 6 times is really 600 times the multiplicand, the combination 2056914 having been removed two places to the left, and therefore made 100 times greater in value; and what appears to be 5 times is really 50,000 times the multiplicand, the combination 1714095 having been removed four places to the left, and therefore made 10,000 times greater. In fact, we have broken up the multiplier into 7, 600, and 50,000—the parts represented by the digits 7, 6, and 5, respectively; and we have added together, for the required product, 7 times, 600 times, and 50,000 times the multiplicand. Written in full, the product of the multiplicand by 600, which is resolvable into the factors 6 and 100, would be (342819 × 6 = 2056914; 2056914 × 100 =) 205,691,400; and the product of the multiplicand by 50,000, which is resolvable into the factors 5 and 10,000, would be (342819 × 5 = 1714095; 1714095 × 10,000 =) 17,140,950,000. So that, in its uncontracted form, the work would stand as in the margin.

342819
50607
2399733
205691400
17140950000
17349041133

58. Rule for Simple Multiplication, when the multiplier is greater than 12, and is neither a power of 10 nor a number resolvable into factors: Write the multiplier under the multiplicand, so that every two figures belonging to the same place shall stand in the same vertical column. Then, beginning at the right-hand side, multiply the multiplicand by the digits of the multiplier successively, and set down the resulting products—one under the other—in such a way that the most right-hand figure of each shall stand in the same vertical column as the digit which, as multiplier, produced it. Add together the results so obtained, and the sum will be the required product.

[Multiplication of Decimals.]

Case IV., in which a decimal occurs in the multiplicand, or in the multiplier, or in both.

Example VII.—Multiply each of the following numbers by 234:—678·9; 67·89; and 6·789.

Here it is to be observed, before we proceed further, that when, in an exercise in multiplication, we employ the true multiplier and too large a multiplicand, we obtain too large a product; and that, in such cases, the product is always as many times too large as the multiplicand is. When, for instance, we multiply 5 by 3, we obtain a certain product (15); when we multiply 10, the double of 5, by 3, we obtain twice as large a product (30); when we multiply 15, the treble of 5, by 3, we obtain 3 times as large a product (45); and so on:—

5	10	15	20	25	30
3	3	3	3	3	3
15	30	45	60	75	90

Returning to Ex. VII., we multiply as if there were no decimal, and obtain for product (6789×234=) 1588626. [See Ex. V.] This product is 10 times larger than the first, 100 times larger than the second, and 1,000 times larger than the third of the required products: 6789 being 10 times larger than 678·9, 100 times larger than 67·89, and 1,000 times larger than 6·789. Making the combination 1588626 *ten* times smaller in value, therefore, by removing the decimal point *one* place to the

left, we find 678·9×234=158862·6; making 1588626 *one hundred* times smaller, by removing the decimal point *two* places to the left, we find 67·89×234=15886·26; and making 1588626 *one thousand* times smaller, by removing the decimal point *three* places to the left, we find 6·789×234=1588·626:—

6789×234=1588626; 678·9×234=158862·6; 67·89×234=15886·26; 6·789×234=1588·626.

EXAMPLE VIII.—Multiply 6789 by 23·4, by 2·34, and by ·234.

Here it is to be observed that when we employ the true multiplicand and too large a multiplier, we obtain a product as many times too large as the multiplier is. Thus, when we multiply 5 by 3, we obtain a certain product (15); when we multiply 5 by 6, the double of 3, we obtain twice as large a product (30); when we multiply 5 by 9, the treble of 3, we obtain 3 times as large a product (45); &c.:—

5	5	5	5	5	5
3	6	9	12	15	18
15	30	45	60	75	90

Returning to Ex. VIII., we multiply as if there were no decimal, and obtain for product (6789×234=) 1588626. This product is 10 times larger than the first, 100 times larger than the second, and 1,000 times larger than the third of the required products: 234 being 10 times larger than 23·4, 100 times larger than 2·34, and 1,000 times larger than ·234. Proceeding as in Ex. VII., therefore, we make the combination 1588626 *ten* times, *one hundred* times, and *one thousand* times smaller in value, respectively, and find—

6789×23·4=158862·6; 6789×2·34=15886·26; and 6789×·234=1588·626.

EXAMPLE IX.—Multiply 67·89 by 23·4.

We have already seen [Ex. VII.] that the product of 67·89 and 234 is 15886·26; therefore, as 23·4 is 10 times smaller than 234, the required product must be 10 times smaller than 15886·26. So that, removing the decimal point in the combination 15886·26 *one* place to the left, we find 67·89×23·4=1588·626.

EXAMPLE X.—Multiply 678·9 by 2·34.

The product of 678·9 and 234 [Ex. VII.] is 158862·6; therefore, 2·34 being 100 times smaller than 234, the required product must be 100 times smaller than 158862·6. So that, removing the decimal point in the combination 158862·6 *two* places to the left, we find 678·9×2·34=1588·626

Comparing the results obtained in the preceding examples, we see that, in every case, the decimal point is removed as far to the left in the combination 1588626 as there are decimal places in the factors—it being immaterial whether those decimal places occur exclusively in the multiplicand, or exclusively in the multiplier, or partly in the one and partly in the other:—

$$6789 \times 234 = 1588626$$

$$\left.\begin{array}{r} 678{\cdot}9 \times 234 \\ \text{or } 6789 \times 23{\cdot}4 \end{array}\right\} = 158862{\cdot}6 \qquad \left.\begin{array}{r} 6{\cdot}789 \times 234 \\ \text{or } 6789 \times {\cdot}234 \\ \text{,, } 67{\cdot}89 \times 23{\cdot}4 \\ \text{,, } 678{\cdot}9 \times 2{\cdot}34 \end{array}\right\} = 1588{\cdot}626$$

$$\left.\begin{array}{r} 67{\cdot}89 \times 234 \\ \text{,, } 6789 \times 2{\cdot}34 \end{array}\right\} = 15886{\cdot}26$$

59. Rule for Simple Multiplication, when a Decimal occurs in the multiplicand, or in the multiplier, or in both: Multiply as if there were no Decimal, and remove the decimal point in the resulting product as many places to the left as there are decimal places in the multiplicand and the multiplier.

60. *Proof of Multiplication.*—An exercise in Multiplication is presumed to have been correctly performed (1.) when, on taking the multiplier for multiplicand, and the multiplicand for multiplier, we obtain the same product as before; or (2.) when, on dividing the product by the multiplier, we obtain the multiplicand for quotient. A third proof—known as the "Casting out of the *Nines*"—is this: Take 9 as often as possible from the sum of the digits* of the multiplicand, and also from the sum of the digits of the multiplier; multiply the two remainders together, and take 9 as often as possible from the sum of the digits of the resulting product; if what is then left be the same as what is left when 9 is taken as often as possible from the sum of the digits of the "answer," the work is probably correct.

The first of these proofs—applicable to both Simple and Fractional Multiplication—is based upon the fact that, in finding the product of two numbers, we are at liberty to take either of

* By "the sum of the digits" is meant, of course, the sum of the value which the digits would have in the *units' place.*

the two for multiplicand, and the other for multiplier. The second proof—applicable to Simple, Compound, and Fractional Multiplication alike—is based upon the fact that, when the product of two factors is divided by one of them, the other factor should be obtained for quotient.* The third proof—applicable to Simple Multiplication only—depends upon the fact that, when 9 is taken as often as possible from the sum of the digits of a (simple) number, what remains is the same as what would be left if 9 were taken as often as possible from the number itself.

This last fact is easily established. The number 6782, for instance, can be written 6,000+700+80+2. Now—

$$\begin{array}{rlll} 6{,}000 = 6\times & 1{,}000 = 6\times(999+1) & = 6\times 999+6 \\ 700 = 7\times & 100 = 7\times(\ \ 99+1) & = 7\times\ \ 99+7 \\ 80 = 8\times & 10 = 8\times(\ \ \ \ 9+1) & = 8\times\ \ \ \ 9+8 \\ 2 = & & \qquad\qquad\ \ 2 \end{array}$$

As 999 contains an exact number of nines, so does 6×999. As 99 contains an exact number of nines, so does 7×99. And of course 8×9 contains an exact number of nines. It is obvious, therefore, that whatever would remain after the subtraction of 9 as often as possible from the given number, is what would remain after the subtraction of 9 as often as possible from $6+7+8+2$, the sum of the digits of the given number; the remaining portions $(6\times 999+7\times 99+8\times 9)$ of the number consisting, as we have seen, of an exact number of nines. It could be shown, in the same way, that the subtraction of 9 as often as possible from 534 would leave the same remainder as would be left if 9 were subtracted as often as possible from $5+3+4$,—the sum of the digits of 534.

Let us now take 6,782 for multiplicand, and 534 for multiplier. Subtracting 9 as often as possible from the sum of the digits of the multiplicand, we have 5 remaining:—$6+7=13$; $13-9=4$; $4+8=12$; $12-9=3$; $3+2=5$. Consequently, 5 would remain if 9 were subtracted as often as possible from the multiplicand itself. So that the multiplicand contains a certain number of nines, and 5 over. If, therefore, 5 be taken from the multiplicand, there will remain an exact number of nines, under the form $(6782-5=)$ 6777. As $6782=6777+5$, 534×6782 (i.e., the required product) $=534\times(6777+5)=534\times 6777+534\times 5$. And as 6777 contains an exact number of nines, so does 534×6777. Whatever, therefore, would remain after

$$\begin{array}{r} 6782 \\ 534 \\ \hline 27128 \\ 20346 \\ 33910 \\ \hline 3621588 \end{array}$$

* Of course this proof supposes a knowledge of Division—Simple, Compound, or Fractional Division, as the case may be.

the subtraction of 9 as often as possible from the required product, is what would remain after the subtraction of 9 as often as possible from 534×5. We next find that when 9 is taken as often as possible from the sum of the digits of the multiplier, 3 remains:—$5+3=8$; $8+4=12$; $12-9=3$. Consequently, 3 would remain if 9 were taken as often as possible from the multiplier itself. In other words, the multiplier contains a certain number of nines, and 3 over. When, therefore, we subtract 3 from 534, we have an exact number of nines remaining, under the form $(534-3=)$ 531. As $534=531+3$ $534 \times 5=(531+3) \times 5=531 \times 5+3 \times 5$. And as 531 contains an exact number of nines, so does 531×5. Whatever, therefore, would remain after the subtraction of 9 as often as possible from 534×5, is what would remain if 9 were taken as often as possible from 3×5.

It follows, then, that whatever would remain after the subtraction of 9 as often as possible from the required product, is what would remain if 9 were taken as often as possible from 3×5. But subtracting 9 as often as possible from the required product is the same—so far as the remainder is concerned—as subtracting 9 as often as possible from the sum of the digits of the product. We therefore conclude that the product has been correctly set down when, on taking 9 as often as possible from the sum of its digits, we obtain the same remainder as is obtained when 9 is taken as often as possible from 3×5, or—what amounts to the same thing—from the sum of the digits of 15, the product of 3 and 5. And it will be observed that 5 and 3 are the remainders obtained when 9 is taken as often as possible from the sum of the digits of the multiplicand and the sum of the digits of the multiplier, respectively.—Taking 9 as often as possible from the sum of the digits of $(3 \times 5=)$ 15, we have $(1+5=)$ 6 remaining, and this is also the remainder when 9 is taken as often as possible from the sum of the digits of the "answer":—$3+6=9$; $9-9=0$; $2+1=3$; $3+5=8$; $8+8=16$; $16-9=7$; $7+8=15$; $15-9=6$.

It is scarcely necessary to remark that, as a test of accuracy, the "Casting out of the Nines" would be valueless if the digits of the product happened to be written in wrong places, or if the product contained ciphers which did not belong to it, or, again, if there were a "balance of opposite errors"—a 7, for example, being substituted for a 5 in one place, and a 4 for a 6 in another. Such mistakes, however, are seldom committed by a person who is ordinarily careful, and whose object is, not to mislead, but to obtain the true product: so that the "Casting out of the Nines," although by no means an infallible test, is not at all as worthless as some writers on Arithmetic appear to consider it.

SIMPLE DIVISION.

61. DIVISION is an operation by which we find how many times one number, called the *divisor*, can be taken from—or is contained as addend in—another number, called the *dividend*.

62. In an exercise in Division, the number indicating how often the divisor can be taken from the dividend is termed the *quotient*. Any portion that may remain of the dividend, after the subtraction of the divisor as often as possible, is termed the *remainder*, which, it is obvious, must always be less than the divisor.

Here, for instance, is an exercise in Division: How many persons could be paid 4 shillings each, out of a purse containing 31 shillings?

Performing the series of subtractions exhibited in the margin, we see that the purse-bearer would have 27 shillings remaining after paying the first person, 23 shillings after paying the second person, 19 shillings after paying the third person, and so on, until, after paying the seventh person, the purse-bearer would have only 3 shillings left.

Here is another exercise in Division: If 31 shillings were divided equally amongst 4 persons, what would each person's share be?

Proceeding as before, we see that, after payment of a shilling to each person, 27 shillings would remain; that, after payment of a second shilling to each person, 23 shillings would remain; that, after payment of a third shilling to each person, 19 shillings would remain; and so on, until, after each person had received a shilling 7 different times,—that is, had received 7 shillings altogether,—only 3 shillings would be left.

$$
\begin{array}{rl}
31 & \\
\underline{4} & (1) \\
27 & \\
\underline{4} & (2) \\
23 & \\
\underline{4} & (3) \\
19 & \\
\underline{4} & (4) \\
15 & \\
\underline{4} & (5) \\
11 & \\
\underline{4} & (6) \\
7 & \\
\underline{4} & (7) \\
3 &
\end{array}
$$

So that the preceding questions both resolve themselves into this: How often could 4 shillings be taken from 31 shillings? Knowing the multiplication table, we are able to answer this question without having recourse to subtraction. For, remembering that 7 fours are 28, and that 8 fours are 32, we see that there are 7 fours, but not 8 fours in 31; and that, if 7 fours were taken (or, in other words, if 4 were taken 7 times) from 31, 3 would remain. Here 4 is the "divisor," 31 the "dividend," 7 the "quotient," and 3 the "remainder."

$$
\begin{array}{r}
4)\underline{31} \\
7+3
\end{array}
$$

63. Division is called SIMPLE when the divisor, the dividend, and the resulting quotient are simple numbers.*

64. When the dividend is a concrete number, the divisor and the quotient are—one a concrete number of the same denomination as the dividend, and the other an abstract number. In no case can the divisor and the quotient be *both* concrete. When the dividend is an abstract number, both the divisor and the quotient are abstract also. It is scarcely necessary to add that (when there is one) the remainder—being a portion of the dividend—is concrete or abstract according as the dividend is concrete or abstract.

Thus, the division of 31*s.* by 4 might mean (1.) the separation of 31*s.* into shares of 4*s.* each, or (2.) the separation of 31*s.* into 4 equal shares. In the former case, the divisor (4*s.*)—indicating the value of a share—would be concrete, whilst the quotient (7)—indicating the number of shares—would be abstract. In the latter case, the divisor (4)—indicating the number of shares—would be abstract, whilst the quotient (7*s.*)—indicating the value of a share—would be concrete. In either case, the remainder (3*s.*) would obviously be concrete, and of the same denomination as the dividend. If, however, instead of 31*s.* we took the abstract number 31 for dividend, the divisor, quotient, and remainder would *all* be abstract.

Mental Exercises in Simple Division.

Under this head would come such questions as the following—the divisor in no case exceeding 12, and the dividend in no case exceeding 12 times the divisor:

If 40 trees were planted in rows of 5 each, how many rows would there be? How long would a person, travelling at the rate of 7 miles a day, be performing a journey of 63 miles? A quantity of beef, weighing 108 pounds, was divided equally amongst 9 persons; how much did each person get? How many *sixes* in 57? How many *eights* in 90? How many *elevens* in 130? &c., &c., &c.

---

* The division of 31*s.* by 4, for instance, is SIMPLE when we write 7 for quotient and 3 for remainder; the division is COMPOUND when—the remainder, 3*s.*, being reduced to 36*d.*, and then divided by 4—we set down 7*s.* 9*d.* for quotient; and the division is FRACTIONAL when we write $7\frac{3}{4}$*s.* for quotient.

SLATE EXERCISES IN SIMPLE DIVISION
are most conveniently dealt with in the following order:—

CASE I., in which the divisor is a power of 10; or in which the divisor does not exceed 12, but the dividend exceeds 12 times the divisor.

EXAMPLE I.—Divide 365 by 100.*

Here we simply remove the decimal point (supposed to stand after the units' figure, 5) two places to the left—that is, as many places as there are ciphers in the combination 100. We thus find the required quotient to be 3·65.—Instead of setting down 3·65 for quotient, we may, if we please, take 3 for quotient and 65 (units) for remainder: on the principle that 100 is contained 3 times in 365, and that, if 100 were taken 3 times from 365, 65 would remain.

EXAMPLE II.—Divide 789 by 4.

In this case the dividend, on account of its largeness, must be broken up into the parts which its digits individually represent—namely, 7 groups of 100 each, 8 groups of 10 each, and 9 units. Let us suppose the dividend to be a sum of money which we want to divide equally between 4 persons, and that the amount consists of 7 *one-hundred*-pound notes, 8 *ten*-pound notes, and 9 *one*-pound notes. Of the *one-hundred*-pound notes we give each person 1: the number of such notes is then diminished by $(1\times4=)$ 4, and we have $(7-4=)$ 3 remaining. Those 3 we change into $(3\times10=30)$ *ten*-pound notes, of which we then have, altogether, $(30+8=)$ 38. Giving each person 9 of the 38 *ten*-pound notes, we diminish the number of such notes by $(9\times4=)$ 36, and have $(38-36=)$ 2 remaining. Those 2 we convert into $(2\times10=20)$ *one*-pound notes, of which we then have, altogether, $(20+9=)$ 29. Giving each person 7 of the 29 *one*-pound notes, we diminish the number of such notes by $(7\times4=)$ 28, and have $(29-28=)$ 1 remaining. So that, altogether, each person receives 197 pounds; and after payment of this amount to each person, 1 pound remains. In other words, the division of 789 by 4 gives 197 for quotient, and 1 for remainder.

$$\begin{array}{r} 4)789 \\ \hline 197+1 \end{array}$$

This operation would be more troublesome if we began (as we do in Addition, Subtraction, and Multiplication) at the right-hand side. Thus, dividing the 9 units by 4, we should obtain 2 units for quotient, and 1 for remainder. Dividing the 8 tens by 4, we should obtain 2 tens for quotient (with no remainder). Dividing the 7 hundreds by 4, we should obtain 1

$$\begin{array}{l} 789(4 \\ \hline 1\tfrac{3}{4}\tfrac{2}{4}+1 \\ \;\;97 \end{array}$$

* Such examples as this might be given as *mental* exercises.

hundred for quotient, and 3 hundreds for remainder. Converting those 3 hundreds into 30 tens, and dividing by 4, we should obtain 7 tens for quotient, and 2 tens for remainder; so that in the tens' place of the answer we should have to write $(2+7=)$ 9, instead of 2. Converting the last remainder, 2 tens, into 20 units, and dividing by 4, we should obtain 5 units for quotient (with no remainder); so that in the units' place of the answer we should have to write $(2+5=)$ 7, instead of 2.

EXAMPLE III.—Divide 102,354 by 12.

Beginning at the left-hand side, we see that neither the first figure (1), nor the first two (10) taken together, are divisible by 12. We therefore take, for our first "partial" dividend, as it is called, 102 groups of a thousand each. Dividing this number by 12, we obtain for quotient 8 groups of a thousand each, and for remainder 6 such groups:—$8 \times 12 = 96$; $102 - 96 = 6$. Converting this remainder into (60) groups of a hundred each, we then have, altogether, $(60+3=)$ 63 such groups. Dividing this number, which constitutes the second partial dividend, by 12, we obtain for quotient 5 groups of a hundred each, and for remainder 3 such groups:—$5 \times 12 = 60$; $63 - 60 = 3$. Converting this second remainder into (30) groups of ten each, we then have, altogether, $(30+5=)$ 35 such groups. Dividing this number, which constitutes the third partial dividend, by 12, we obtain for quotient 2 groups of ten each, and for remainder 11 such groups:—$2 \times 12 = 24$; $35 - 24 = 11$. Converting this last remainder into (110) units, we then have, altogether, $(110+4=)$ 114 units. Dividing this number, which constitutes the fourth and last partial dividend, by 12, we obtain 9 units for quotient, and 6 for remainder:—$9 \times 12 = 108$; $114 - 108 = 6$. We thus find that 12 is contained 8,529 times in 102,354, and that, if 12 were taken 8,529 times from 102,354, 6 (units) would be left.

$$\begin{array}{r} 12)\underline{102354} \\ 8529+6 \end{array}$$

It will be seen that, except for the sake of explanation, it is unnecessary, in the preceding exercise, to speak of "groups of a thousand each," "groups of 100 each," &c.: it being sufficient to regard each partial dividend, in turn, as representing so many units, and to set down each quotient figure in the same place as the most right-hand figure of the partial dividend from which it was obtained.

65. Rule for Simple Division, (1.) when the divisor is a power of 10: Remove the decimal point in the dividend as many places to the left as there are ciphers in the divisor. Take the resulting number for quotient, in which case there will be no "re-

mainder;" or, take the portion to the left of the decimal point for quotient, and the remaining portion, regarded as a whole number, for remainder.—(2.) When the divisor is not a power of 10: Beginning at the left-hand side of the dividend, take for the first partial dividend as many figures as there are in the divisor, and one more if necessary. Divide this partial dividend by the divisor, and set down the quotient figure in the same place as the most right-hand figure of the partial dividend. Then, multiply the divisor by this quotient figure; subtract the product from the partial dividend; and to the remainder annex—or conceive to be annexed—the next figure of the dividend. Take, as the second partial dividend, the number thus obtained, and proceed as before—regarding each partial dividend, in turn, as having its most right-hand figure in the units' place; treating each quotient figure, in turn, (when multiplying the divisor by it,) as the units' figure also; but taking care, at the same time, to write each quotient figure in the same place as the most right-hand figure of the partial dividend which produced it. Continue the process until the units' figure of the quotient has been obtained, and also the "remainder"—if there should be one.

NOTE.—When the divisor is not contained an exact number of times in the dividend, the finding of the units' figure of the quotient does not necessarily terminate the division. For, as thousands are converted into hundreds, hundreds into tens, and tens into units—so, units can be converted into tenths, tenths into hundredths, hundredths into thousandths, &c. Instead, therefore, of setting aside, as "remainder," the units which are left after the finding of the units' figure of the quotient, we can, if we please, continue the division—filling up the decimal places of the quotient successively, until nothing remains, or until the remainder has become so small as to be unworthy of notice. Thus, the division of 102,354 by 12 [Ex. III.] gives 8,529 for quotient, and 6 (units) for remainder. This remainder is convertible into ($6 \times 10 =$) 60 tenths, the division of which by 12 gives 5 tenths for quotient, with no remainder. So that the quotient might be written 8,529·5. Again: the division of 789

by 4 [Ex. II.] gives 197 for quotient, and 1 (unit) for remainder. This remainder is convertible into 10 tenths, the division of which by 4 gives 2 tenths for quotient, and 2 tenths for remainder. This last remainder is convertible into $(2\times 10=)$ 20 hundredths, the division of which by 4 gives 5 hundredths for quotient, with no remainder. So that the quotient might be written 197·25. As a third illustration, let it be required to divide 453 by 7. The first partial dividend is 45 tens, from which we obtain 6 tens for quotient, and 3 tens for remainder. Annexing the 3 units to this remainder, we have, as the second partial dividend, 33 units, from which we obtain 4 units for quotient, and 5 units for remainder. This second remainder, by annexing a cipher, we convert into $(5\times 10=)$ 50 tenths, from which we obtain 7 tenths for quotient, and 1 tenth for remainder. This third remainder, by annexing a cipher, we convert into 10 hundredths, from which we obtain 1 hundredth for quotient, and 3 hundredths for remainder. This fourth remainder, by annexing a cipher, we convert into $(3\times 10=)$ 30 thousandths, from which we obtain 4 thousandths for quotient, and 2 thousandths for remainder. Rejecting this last remainder, which is extremely small, we thus obtain for quotient 64·714. If the three dividends—102354, 789, and 453—were written under the forms 102354·0, 789·00, and 453·000, respectively, the work would stand thus:

12)102354·0	4)789·00	7)453·000
8529·5	197·25	64·714

Case II., in which the divisor exceeds 12, but is resolvable into a pair of factors—each of them either a power of 10 or a number not greater than 12.

Example IV.—How many persons could be paid 36 pounds each, out of a purse containing 9,574 pounds?

This question involves the division of 9,574 by 36. Resolving 36 into the factors 12 and 3, and dividing 9,574 by 12, we see that in 9,574 pounds there are 797 sums of 12 pounds each, and 10 pounds besides. Now, as often as 3 sums of 12 pounds each could be taken from 797 such sums, so often could 36 pounds be taken out of the purse. Dividing 797 by 3, therefore, we find the required number of persons to be 265. The second division leaves 2 for remainder—that is, 2 (of the 797) sums of 12 pounds each: in other words, $(2\times 12=)24$ pounds. To this amount we add 10

$$\begin{array}{l} 12)\underline{9574} \\ 3)\underline{797+10} \\ \quad 265+(2\times 12+10=)34 \end{array}$$

pounds, the first remainder, and we thus find the "true" remainder, as it is called, to be (24+10=)34 pounds.—So that the division of 9,574 by 36 gives 265 for quotient, and 34 for remainder.

Let us next take the factors 9 and 4. Dividing by 9, we see that the purse contains 1,063 sums of 9 pounds each, and 7 pounds besides. So that the question resolves itself into this: How often could 4 sums of 9 pounds each be taken from 1,063 such sums? Because, so often could 36 pounds be taken out of the purse. We therefore divide 1,063 by 4. and find the required number of persons to be, as before, 265. In this case the second division leaves 3 for remainder—that is, 3 (of the 1,063) sums of 9 pounds each: in other words, (3×9=)27 pounds. Adding 7 pounds, the first remainder, to this amount, we find the true remainder to be, as before, (27+7=)34 pounds.

$$\begin{array}{l} 9)\underline{9574} \\ 4)\underline{1063+7} \\ \quad 265+(3\times9+7=)34 \end{array}$$

Example V.—If 458,279 apples were formed into heaps of 7,000 each, how many heaps would there be?

This question involves the division of 458,279 by 7,000. Resolving 7,000 into the factors 1,000 and 7, and dividing 458,279 by 1,000, we see that, if the apples were formed into heaps of 1,000 each, there would be 458 such heaps, and 279 apples besides. As 7 heaps of 1,000 each would be required to form one heap of 7,000, the question now resolves itself into this: How often could 7 be taken from 458? Because, so often could one of the larger heaps be formed. Dividing 458 by 7, therefore, we find the required number of heaps of 7,000 each to be 65. The second division gives 3 for remainder—that is, 3 heaps of 1,000 each: in other words, (3×1,000=)3,000. To this we add the first remainder, 279, and the true remainder is thus found to be (3,000+279=) 3,279.

$$\begin{array}{l} 1,000)\underline{458279} \\ \qquad 7)\underline{458+279} \\ \qquad\quad 65+(3\times1,000+279=)3,279. \end{array}$$

66. Rule for Simple Division, when the divisor exceeds 12, but is resolvable into a pair of factors—each of them either a power of 10 or a number not greater than 12: Divide the dividend by one of the factors of the divisor, and the resulting quotient by

the other factor. The second quotient will be the true quotient. To find the true remainder—multiply the first divisor by the second remainder, and to the product add the first remainder.

Case III., in which the divisor exceeds 12, and is not resolvable into factors.

Example VI.—Divide 23,456 by 89.

Let us suppose the dividend to be a sum of money, which we want to divide equally between 89 persons; and that the amount consists of 6 *one*-pound notes, 5 *ten*-pound notes, 4 *one-hundred*-pound notes, 3 *one-thousand*-pound notes, and 2 *ten-thousand*-pound notes. Of the *ten-thousand*-pound notes we can give each person none, there being only 2 such notes altogether. These 2, therefore, we change into (20) *one-thousand*-pound notes, of which we then have, altogether, (20+3=)23. Even of these we can give each person none. We therefore change them into (230) *one-hundred*-pound notes, of which we then have, altogether, (230+4=)234. Of these we give each person 2. The number of *one-hundred*-pound notes is then diminished by (89×2=)178, and we have (234−178=)56 remaining. Converting this remainder into (560) *ten*-pound notes, we then have, altogether, (560+5=)565 such notes, of which each person gets 6. The number of *ten*-pound notes is then diminished by (89×6=)534, and we have (565−534=)31 remaining. Converting this remainder into (310) *one*-pound notes, we then have, altogether, (310+6=)316 such notes, of which each person gets 3. The number of *one*-pound notes is then diminished by (89×3=)267, and we have (316−267=)49 remaining. So that when 23,456 pounds are divided equally between 89 persons, each person receives, as his share, 263 pounds, and 49 pounds are left after the distribution. In other words, the division of 23,456 by 89 gives 263 for quotient, and 49 for remainder.

```
     263
89)23456
   178
   ---
    565
    534
    ---
     316
     267
     ---
      49
```

We could, if it were necessary, proceed further in this case. Thus, converting the 49 units into (49×10=)490 tenths, and dividing by 89, we should obtain 5 tenths for quotient, and 45 tenths for remainder:—89 × 5=445; 490 −445=45. Converting this remainder into (45×10=)450 hundredths, and dividing by 89, we should next obtain 5

hundredths for quotient, and 5 hundredths for remainder:— $89 \times 5 = 445$; $450 - 445 = 5$. This last remainder—which, even when converted into 50 thousandths, is not divisible by 89—being rejected on account of its smallness, the quotient in its extended form would thus be found to be 263·55. If the dividend were written 23456·00, the work would stand as in the margin.

$$
\begin{array}{r}
263{\cdot}55 \\
89)\overline{23456{\cdot}00} \\
\underline{178}\phantom{456{\cdot}00} \\
565\phantom{6{\cdot}00} \\
\underline{534}\phantom{6{\cdot}00} \\
316\phantom{{\cdot}00} \\
\underline{267}\phantom{{\cdot}00} \\
490 \\
\underline{445} \\
450 \\
\underline{445} \\
5
\end{array}
$$

It will be seen that in this last exercise, which belongs to what school-boys call Long Division, we proceed exactly as in Examples II. and III., which belong to what is termed Short Division. In each of the three instances we break up the dividend into two or more "partial" dividends, which we divide successively by the divisor—multiplying the divisor by the several quotient figures in succession; subtracting each of the resulting products from the corresponding partial dividend; and annexing to each remainder (except the last) the next figure of the dividend, in order to form a new partial dividend. The only difference is, that in Short Division the multiplications and subtractions are performed *mentally*, whilst in Long Division they are not: for which reason the quotient, in Long Division, cannot be written—as it is in Short Division—*under* the dividend. In Long Division the quotient is usually placed to the right of the dividend; but why, in ordinary cases, should it not be written *over* the dividend—that the pupil may the more distinctly see the place to which each quotient figure belongs, and may be saved the trouble of "marking" the figures of the dividend according as they are annexed or "brought down"?* At all events, the Rule for Long Division is exactly the same as that already given for Short Division. [See § 65, part 2.]

[DIVISION OF DECIMALS.]

CASE IV., in which a decimal occurs in the dividend, or in the divisor, or in both.

EXAMPLE VII.—Divide 234·56 by 89.

Neither the first figure (2) of the dividend, nor the first two (23) taken together, being divisible by 89, we take, as the first

* Two operations are performed by the "bringing down" of a figure from the dividend: the remainder is multiplied by 10, and the brought-down figure added to the product.

partial dividend, 234 units. Dividing this number by 89, we obtain 2 units for quotient, and 56 units for remainder:—$89 \times 2 = 178$; $234 - 178 = 56$. Converting this remainder into ($56 \times 10 = 560$) tenths, we then have ($560 + 5 =$)565 tenths altogether. The division of this number by 89 gives 6 tenths for quotient, and 31 tenths for remainder:—$89 \times 6 = 534$; $565 - 534 = 31$. Converting this remainder into ($31 \times 10 = 310$) hundredths, we then have ($310 + 6 =$)316 hundredths altogether. Dividing this number by 89, we obtain 3 hundredths for quotient, and 49 hundredths for remainder: —$89 \times 3 = 267$; $316 - 267 = 49$. So that the required quotient is 2·63.

$$\begin{array}{r} 2{\cdot}63 \\ 89)\overline{234{\cdot}56} \\ 178 \\ \hline 565 \\ 534 \\ \hline 316 \\ 267 \\ \hline 49 \end{array}$$

EXAMPLE VIII.—Divide 23·456 by 89.

Here we take, as our first partial dividend, 234 tenths. The division of this number by 89 gives 2 tenths for quotient, and 56 tenths for remainder:—$89 \times 2 = 178$: $234 - 178 = 56$. Converting this remainder into ($56 \times 10 = 560$) hundredths, we then have, altogether, ($560 + 5 =$)565 hundredths. The division of this number by 89 gives 6 hundredths for quotient, and 31 hundredths for remainder:—$89 \times 6 = 534$; $565 - 534 = 31$. Converting this remainder into ($31 \times 10 = 310$) thousandths, we then have, altogether, ($310 + 6 =$)316 thousandths Dividing this number by 89, we obtain 3 thousandths for quotient, and 49 thousandths for remainder:—$89 \times 3 = 267$; $316 - 267 = 49$. So that the required quotient is ·263.

$$\begin{array}{r} {\cdot}263 \\ 89)\overline{23{\cdot}456} \\ 17\ 8 \\ \hline 5\ 65 \\ 5\ 34 \\ \hline 316 \\ 267 \\ \hline 49 \end{array}$$

EXAMPLE IX.—Divide 2·3456 by 89

In this case, the first partial dividend is 234 hundredths. Dividing this number by 89, we obtain 2 hundredths for quotient, and 56 hundredths for remainder:—$89 \times 2 = 178$; $234 - 178 = 56$. Converting this remainder into ($56 \times 10 = 560$) thousandths, we then have, altogether. ($560 + 5 =$) 565 thousandths. Dividing this number by 89, we obtain 6 thousandths for quotient, and 31 thousandths for remainder: — $89 \times 6 = 534$; $565 - 534 = 31$. Converting this remainder into ($31 \times 10 = 310$) ten-thousandths, we then have, altogether, ($310 + 6 =$)316 ten-thousandths. Dividing this number by 89, we obtain 3 ten-thousandths for quotient, and 49 ten-thousandths for remainder:—$89 \times 3 = 267$; $316 - 267 = 49$. So that the required quotient is ·0263.

$$\begin{array}{r} {\cdot}0263 \\ 89)\overline{2{\cdot}3456} \\ 1\ 78 \\ \hline 565 \\ 534 \\ \hline 316 \\ 267 \\ \hline 49 \end{array}$$

Comparing the last three examples with Example VI., we see that the whole four are worked in exactly the same way. In every case the partial dividends are 234, 565, and 316, respectively, and are regarded—each in its turn—as representing so many units. Because, in working the examples mechanically, we dispense with the words "tenths," "hundredths," "thousandths," &c., as well as with "tens," "hundreds," "thousands," &c.—such words being employed only when we want to explain what we are doing.

67. Rule for Simple Division, when a Decimal occurs in the Dividend, but not in the Divisor: Divide as if there were no Decimal, and write the decimal point in the resulting quotient in such a position that each quotient figure shall stand in the same place as the most right-hand figure of the partial dividend which produced it.

Before we proceed further, it is to be observed that, instead of dividing a given dividend by a given divisor, we may, if we please, divide the double of the dividend by the double of the divisor, or 3 times the dividend by 3 times the divisor, or—in a word—*any* number of times the dividend by the *same* number of times the divisor: the quotient being the same in the one case as in the other. Thus, dividing 12 by 3 is the same—so far as the quotient is concerned—as dividing the double of 12 by the double of 3, or 3 times 12 by 3 times 3, or *any* number of times 12 by the *same* number of times 3.

3)12(4
6)24(4
9)36(4
12)48(4
30)120(4
300)1200(4

Example X.—Divide 23·456 by 8·9.

Here we multiply both the divisor and the dividend by 10, in order to have a whole number for divisor: 8·9×10=89; 23·456×10=234·56. We then divide 234·56 by 89, and find the required quotient to be 2·63. [See Ex. VII.]

Example XI.—Divide ·23456 by ·89.

Here we take 89 for divisor, and remove the decimal point of the dividend two places to the right—that is, we multiply both of the given numbers by 100: ·89×100=89; ·23456×100=23·456. We then divide 23·456 by 89, and find the required quotient to be ·263. [See Ex. VIII.]

Example XII.—Divide ·0023456 by ·089.

Here we take 89 for divisor, and remove the decimal point of the dividend three places to the right—that is, we multiply both

of the given numbers by 1,000: ·089 × 1,000 = 89; ·0023456 × 1,000 = 2·3456. We then divide 2·3456 by 89, and find the required quotient to be ·0263. [See Ex. IX.]

EXAMPLE XIII.—Divide 23456 by ·0089.

Here we take 89 for divisor, and remove the decimal point of the dividend four places to the right—that is, we multiply both of the given numbers by 10,000: ·0089 × 10,000 = 89; 23456 × 10,000 = 234560000. We then obtain the required quotient by dividing 234560000 by 89.

68. Rule for Simple Division, when a Decimal occurs in the Divisor: Regard the Divisor as a whole number, and remove the decimal point of the Dividend as many places to the right as there are decimal places in the Divisor; then, proceed as already directed. [§ 65, part 2; or § 67—as the case may be.]

NOTE.—In applying this rule, the pupil should first set down the given divisor and dividend, and then make the necessary changes underneath—the original pair of numbers being "cut off" by a horizontal line. Thus, in the case of the last two examples:

·089)·0023456	·0089)23456
89)2·3456	89)234560000

69. To *prove* an exercise in Division—whether it be Simple, Compound, or Fractional Division—we multiply the quotient by the divisor, and add the remainder—should there be one—to the product. When the dividend is obtained for result, the work is presumed to have been correctly performed.

The reason of this is obvious. When, for instance, 789 pounds are divided equally between 4 persons [Ex. II.], each person receives 197 pounds, and 1 pound is left. The total amount distributed, therefore, is 4 times 197 pounds—that is, 788 pounds; and adding to this the amount (1 pound) remaining undistributed, we necessarily obtain the original sum:— 197 × 4 = 788; 788 + 1 = 789.

TABLES

OF

MONEY, MEASURE, WEIGHT, &c.

Money.

2 Farthings	=	1 Half-penny
2 Half-pence, or 4 Farthings	=	1 Penny (*d.*)
12 Pence	=	1 Shilling (*s.*)
20 Shillings	=	1 Pound or Sovereign (£.)

The letters £ *s. d.*—denoting pounds, shillings, and pence, respectively—are the initials of the names of three old Roman Coins: "libra," "solidus," "denarius." A smaller amount than a penny is expressed as a fraction of a penny. Thus, a farthing is written ¼*d.*—*one-fourth* of a penny; two farthings, or a half-penny, ½*d.*—*one-half* of a penny; and three-farthings, ¾*d.*—*three-fourths* of a penny.

Although the pound, the shilling, and the penny are the only "coins of account"—that is, the only coins employed in the keeping of accounts, and in ordinary calculations—there are, altogether, 12 coins in circulation: 2 *gold* coins—the sovereign and the half-sovereign; 7 *silver* coins—the crown or five-shilling piece, the half-crown, the florin or two-shilling piece, the shilling, the sixpence, the fourpence, and the three-pence; and three *bronze* coins—the penny, the half-penny, and the farthing.

Besides the twelve coins just mentioned, three silver ones—the two-penny, the three-half-penny, and the penny piece—are occasionally issued from the Mint, by special command of the Queen, for distribution amongst a number of poor persons on the Thursday before Easter—Maundy Thursday, as it is sometimes called in England; but those little coins, which are known as "*Maundy** money," cannot be said to be in circulation, being, in most cases, retained as curiosities by the shop-keepers and other persons into whose hands they pass.

A new sovereign contains 5 dwts. 3¼ grs. of "standard" gold, in every 12 parts of which there are 11 parts of pure gold,

* The word Maundy is said to be derived from *mand*—Saxon for "basket:" because, formerly, it was customary for the English Sovereign to proceed to Whitehall on Maundy Thursday, and distribute alms from a basket.

and 1 part of an alloy consisting of a mixture of silver and copper; so that a Troy pound of standard gold can be coined into $46\frac{29}{40}$ sovereigns (£46 14*s.* 6*d.*), and the Mint price of gold is thus fixed at £3 17*s.* 10½*d.* an ounce.

A new shilling contains 3 dwts. $15\frac{3}{11}$ grs. of standard silver, in every 40 parts of which there are 37 parts of pure silver, and 3 of copper; so that a Troy pound of standard silver can be coined into 66 shillings, and the Mint price of silver is thus fixed at 5*s.* 6*d.* an ounce.

The bronze coins, issued in 1860, are composed of a mixture of copper, zinc, and tin, in every 100 parts of which mixture there are 95 parts of copper, 4 of zinc, and 1 of tin. A bronze penny weighs 145·83 grs.; a bronze half-penny, 87·5 grs.; and a bronze farthing, 43·75 grs.: so that whilst two farthings contain exactly as much bronze as a half-penny, two half-pence contain *more* than a penny. The number of pence to the Avoirdupois pound is 48; of half-pence, (not 96, but) 80; and of farthings, 160. As a substitute for the ordinary Avoirdupois ounce, therefore, we might take 3 pence, or 5 half-pence, or 10 farthings. Moreover, as a substitute for a foot (12 inches), we might take 12 half-pence, placed in contact along a straight edge—a half-penny being exactly an inch in diameter, and 12 half-pence, consequently, representing a foot in length. A foot would also be represented by 4 pence, 4 half-pence, and 4 farthings, properly placed one after the other—a penny and a farthing being exactly 1·2 and ·8 inches, respectively, in diameter.*

Pence are not a legal tender for more than a shilling; nor is silver a legal tender for more than 2 pounds.

The fact does not appear to be very generally known that, with a sufficient number of four-penny and three-penny pieces, a person could, except in three cases, dispense with the bronze coins in paying any amount containing no lower denomination than pence—the exceptional amounts being 1*d.*, 2*d.*, and 5*d.* Thus, 1*s.* 5*d.* could be paid with 2 fourpences and 3 threepences; 2*s.* 1*d.* with 1 fourpence and 7 threepences; 3*s.* 2*d.* with 2 fourpences and 10 threepences; &c. But of course such a sum as 5*s.* 9½*d.*, or 4*s.* 11¼*d.*, or 6*s.* 10¾*d.* could not be paid entirely in silver, the lowest denomination not being pence.

Previously to the year 1825, when the Irish currency was assimilated to the British, the Irish penny was less in value than

* The "old" or "copper" coins, which are being withdrawn from circulation, are composed of pure copper. When new, a copper penny weighed 291⅔ grs., and there were, consequently, 24 copper pence to the Avoirdupois pound. The Mint price of copper was thus fixed at about 2s. a pound (Avoirdupois).

the British penny—13 of the former being worth only 12 of the latter; and although the people of Ireland kept their accounts in "pounds, shillings, and pence," there was no such thing, in the shape of a coin, as an Irish pound or shilling. The Irish "shilling" was merely a name for "12 (Irish) pence," just as the Irish "pound" was a mere name for "20 (Irish) shillings," or "240 (Irish) pence." And as 13 Irish pence were represented by 12 British ones—so, 13 Irish shillings were represented by 12 British shillings, and 13 Irish pounds by 12 British pounds. We therefore diminish a given amount of "Irish" money by its thirteenth part when we want to find the equivalent amount in the present currency. Thus, the thirteenth part of £100 being £7 13*s.* 10*d.*, the subtraction of the latter amount from the former gives £92 6*s.* 2*d.* as the equivalent, in British money, of £100 Irish.

LENGTH.

(*a.*) "*Long*" *Measure.*

12 Inches (in.)	=	1 Foot* (ft.)
3 Feet	=	1 YARD (yd.)
$5\frac{1}{2}$ Yards	=	1 Perch (per.)
40 Perches	=	1 Furlong (fur.)
8 Furlongs	=	1 Mile (m.)

Artisans divide the inch into *eighths*, whilst scientific people divide it into *tenths, hundredths*, &c. Formerly, the inch was divided into 12 equal parts, called *lines*; but this division is seldom—if ever—heard of now, except as a thing of the past.

An "Irish" perch is 7 yards long; and as there are 40 Irish perches in an Irish furlong, and 8 Irish furlongs in an Irish mile, there are ($7\times40\times8=$) 2,240 yards in an Irish mile, which, therefore, is 480 yards longer than an English mile—the number of yards in an English mile being ($5\frac{1}{2}\times40\times8=$) 1,760. A Scotch mile is 1,977 yards in length—that is, 217 yards longer than an English mile. The relative lengths of the three miles might be thus expressed: a Scotch mile, about as long as an English mile and an eighth; an Irish mile, nearly as long as an English mile and a quarter. Eleven Irish are *exactly* equal in length to 14 English miles. A mile originally meant a thousand (*mille*) "paces"—that is, a length shorter than the present English mile by not quite 100 yards.

In the measurement of cloth, calico, linen, ribbons, &c.,

* The Greek foot was 12·14, the Roman foot 11·6, and the old French foot nearly 12·8 inches in length: yet, curiously enough, the average length of the human foot is only 10·3 inches.

drapers divide the yard—not into feet and inches, but—into quarters and nails; thus:

(*b.*) "*Cloth*" *Measure.*

$2\frac{1}{4}$ Inches	=	1 Nail
4 Nails	=	1 Quarter [of a Yard]
4 Quarters	=	1 YARD

An English ell = 5 quarters, or 45 inches; a French ell = 6 quarters, or 54 inches; a Scotch ell = $4\frac{1}{8}$ quarters, or 37 inches; a Flemish ell = 3 quarters, or 27 inches.

The ell is not now a legal measure, having been abolished by Act of Parliament in 1825. Nevertheless, the English ell is still sometimes used by drapers, but chiefly in the measurement of the *widths* of cloths, &c. Drapers also use the word "Piece," to which three different meanings are attached—a Piece of *calico* being 28 yards; a Piece of *muslin*, 10 yards; and a Piece of *Irish linen*, 25 yards.

(*c.*) "*Nautical*" *Measure.*

A Fathom	=	6 Feet
,, Knot ,, Geographical Mile	=	2,025 yards (about $1\frac{1}{7}$ Eng. miles)
,, League	=	3 Geographical Miles

The lengths of cables, cordage, &c., as well as the depths of the sea in different places, are usually expressed in fathoms, of which, strange to say, there are three varieties—a fathom meaning, on board a man-of-war, 6 feet; on board a merchant-vessel, $5\frac{1}{2}$ feet; and on board a fishing-smack, 5 feet. A fathom, however, is generally understood to mean a length of 6 feet. It originally meant the distance between the ends of a person's hands, when—the hands being open—both arms were extended so as to form a straight line.

The term "knot," as applied to a geographical mile, is derived from the circumstance that the line* employed in ascertaining a ship's speed is divided by KNOTS into a number of parts of equal length, which bears to a geographical mile the same ratio that the time of the sand-glass, used in connection with the line, bears to an hour. Observing the number of lengths and half-lengths, &c.—or, to speak more technically, the number of knots and half-knots, &c.—"paid out" during the running down of the glass, the people on board are thus able to say that the ship is going the *same* number of (geographical) miles and half-miles per hour. If, for example, the glass were a minute-glass, the distance between every two knots

* "*Log*-line," as it is called, because of the log attached to the end of it.

n the line would be 33¾ yards, the 60th part of a geographical mile—a minute being the 60th part of an hour; and f 10½ lengths (or "knots") were paid out during the running-down of the glass, the vessel would be said to be going 10½ knots—that is, geographical miles—an hour.

The 360th part of the earth's circumference is called a *degree*, and is equal in length to 60 geographical miles.

"League" (which, even as a nautical term, has become almost obsolete) is said to have been originally applied to a three-mile length of road, and is derived from the Celtic word *liag*—a flag, or flat stone, such as would naturally, in the olden time, have been employed to mark three-mile distances, just as the modern mile-stone is made to mark one-mile distances.

(*d.*) *Miscellaneous Measures of Length.*

A	Palm	=	3 Inches
„	Hand*	=	4 „
„	Span	=	9 „
„	Cubit	=	18 „
„	Military Step	=	2½ Feet
„	„ Pace	=	5 „
„	Surveying Chain (100 Links)	=	22 Yards

It will be found from actual measurement that the *palm* of the hand (the thumb not being taken into account) has an average breadth of about 3 inches; that the *hand* itself (the thumb being taken into account) has an average breadth of about 4 inches; and that a man's *span*—that is, the distance between the end of the thumb and the end of the little finger, when the two are stretched as far apart as possible—averages about 9 inches. A *cubit* (from *cubitus*, Lat. for elbow) originally meant the length of the arm, from the elbow to the end of the middle finger. The Roman cubit was 17·4, and the cubit of the Scriptures a little less than 22 inches.

As might be inferred from such names as "foot," "hand," "palm," &c., different parts of the human body were formerly, and very generally, employed as measures of length.† But, so early as the time of Edward II., a *grain of barley* was

* The height of a horse is always expressed in *hands*. We constantly hear a horse described as so many "hands high."

† Even at the present day, the *middle finger* (supposed to represent an eighth of a yard) is the standard of length almost universally employed by women in the measurement of their muslins, calicoes, &c. Who has not heard the material of a woman's dress spoken of as being so many "*finger-lengths*" too long or too short?

the standard of length in England—the united lengths of 3 "round and dry" barley-corns, "taken from the middle of the ear," constituting an inch; 12 such inches, a foot; 3 such feet, a yard; &c. For a long time past, however, the legal standard of length has been the yard. The modern yard represents, according to some authorities, the length of the arm of Henry I., and according to others, the average circumference of chest of the Anglo-Saxon race. After having been in use for some time, this yard was superseded by the ell of 45 inches (supposed to have been borrowed from the Paris drapers) in the reign of Henry VII., but was re-introduced in the time of Elizabeth.

In 1758, an eminent mathematical artist was directed to construct a standard yard from the most authentic copies then available; and two years afterwards the yard so constructed, and which was marked "Standard Yard—1760," was adopted by Parliament. This yard having been destroyed by the fire which broke out in the Houses of Parliament in 1834, a new standard was constructed of bronze in 1855, and deposited in the office of the Exchequer. Of this standard, several accurate copies are carefully preserved—one at Westminster, one at the Mint, one at the Royal Observatory, and one at the house of the Royal Society.

SURFACE OR AREA.

144	Square Inches (sq. in.)	=	1 Square Foot (sq. ft.)
9	Square Feet	=	1 SQUARE YARD (sq. yd.)
30¼	Square Yards	=	1 Square Perch (sq. per.)
40	Square Perches	=	1 Rood (r.)
4	Roods*	=	1 Acre (a.)
640	Acres*	=	1 Square Mile (sq. m.)

These measures are commonly known—sometimes as "Superficial," and sometimes as "Square or Land" Measures. The area of a country is expressed in square miles; of an estate or a farm, in acres, roods, and (square) perches; and of a quantity of paving, flooring, roofing, painters' work, &c., in square yards, square feet, square inches, and certain subdivisions—to be noticed elsewhere—of the square foot and square inch. As we shall find hereafter, the number of square inches in a square foot is obtained from the multiplication of 12 (the number of linear inches in a linear foot) by itself; the number of square feet in a square yard, from the multiplication of 3 (the number

* It may not be unnecessary to mention that there is no such measure as a *linear* rood or *linear* acre.

of linear feet in a linear yard) by itself; &c.: a square foot being a square described upon a linear foot; a square yard, a square described upon a linear yard; &c. As there are ($8 \times 40 =$) 320 linear perches in a linear mile, there are ($320 \times 320 =$) 102,400 square perches in a square mile. An acre, therefore—being the 640th part of a square mile—contains ($102,400 \div 640 =$) 160 square perches; and a rood—being the 4th part of an acre—($160 \div 4 =$) 40 square perches. A "square chain"—that is, a square whose side is a chain in length—contains one-tenth of an acre; for, a chain being 22 yards long, the area of a square chain is ($22 \times 22 =$) 484 square yards, whilst the number of square yards in an acre is 4,840.

An Irish acre contains 7,840, and a Scotch acre 6,150·4 square yards; so that 121 Irish are exactly equal in area to 196 English acres, whilst 48 Scotch are very nearly equal to 61 English acres.

SOLIDITY.

1,728 Cubic Inches (cub. in.)	=	1 Cubic Foot (cub. ft.)
27 Cubic Feet	=	1 CUBIC YARD (cub. yd.)

Quantities of timber, stone, brickwork, &c. are expressed in cubic yards, cubic feet, and cubic inches. A large quantity of timber is usually expressed in "tons" or "loads": a ton or load of "rough" timber being 40 cubic feet, and of "hewn" timber 50 cubic feet. A large quantity of stone, also, is sometimes expressed in tons: a ton of marble, for example, being 12 cubic feet; a ton of Portland stone, 16 cubic feet; and a ton of Bath stone, 20 cubic feet.

The capacity of a ship—or rather of the portion available for freight—is expressed in "tons" of 100 cubic feet each. For every such ton, a ship is supposed to be able to carry a weight of a ton and a half (30 cwt.), and in some cases 10 per cent. more. Thus, a ship of 1,000 "tons register" has a capacity of ($1,000 \times 100 =$) 100,000 cubic feet, and is supposed to be capable of carrying 1,500 "tons burden," or ($1,500 + \frac{1}{10}$ of $1,500 =$) 1,650 "tons burden"—according as the build of the vessel is "sharp" or "full." Freight is charged for—sometimes by weight, and sometimes by bulk: in the latter case, the charge is so much per "ton" of 40 cubic feet. So that a ton "register" is equal in capacity to $2\frac{1}{2}$ tons of freight.

As we shall learn farther on, the number of cubic inches in a cubic foot is the product obtained when 12 (the number of linear inches in a linear foot) is taken three times as factor—$12 \times 12 \times 12 = 1,728$; whilst the number of cubic feet in a cubic yard is the product obtained when 3 (the number of linear feet in a linear yard) is taken three times as factor—$3 \times 3 \times 3 = 27$.

CAPACITY.

4 Gills (or "Naggins")	=	1 Pint
2 Pints	=	1 Quart
4 Quarts	=	1 GALLON (277·274 cub. inches)
2 Gallons	=	1 Peck
4 Pecks	=	1 Bushel
8 Bushels	=	1 Quarter
5 Quarters	=	1 Load

The pint is commonly divided (not into gills, but) into half-pints and "glasses"—a glass being supposed to contain the half of a gill, or the eighth part of a pint. The peck, bushel, and quarter are used only for what are called "dry goods" (corn, flour, &c.), in the measurement of which the vessel was formerly "heaped" to a height, above the mouth, equal to one-third of the height of the vessel itself. Heaped measure has been abolished by Act of Parliament; and so, also, has been the "corn" gallon of 268·6 cubic inches, as well as the "ale" and "wine" gallons, of 282 and 231 cubic inches, respectively. The Imperial gallon of 277·274 cubic inches, adopted in the reign of Geo. IV., is now the only legal one, and is the standard of capacity for both liquids and dry goods. An Imperial gallon of distilled water weighs 10 lbs. Avoirdupois.

Special Measures of Capacity.

			Gallons.
A Firkin	of Ale or Beer	=	9
„ Kilderkin		=	18
„ Barrel		=	36
„ Hogshead		=	54
„ Butt		=	108
„ Tun		=	216

			Gallons.
A "Pipe" of	Port	=	115
	Teneriffe	=	100
	Marsala	=	93
	Madeira, or Cape	=	92
	Lisbon, or Bucellas	=	117

			Gallons.
An Anker	of Wine	=	10
A Runlet		=	18
„ Tierce		=	42
„ Hogshead		=	63
„ Puncheon		=	84
„ Tun		=	252

		Gallons.
A "Butt" of Sherry	=	108
An "Aum" of Rhenish Wine*	=	30

Apothecaries' Measures.

[Apothecaries divide the Imperial pint into 20 equal parts, called fluid ounces; the fluid ounce into 8 equal parts, called fluid drachms; the fluid drachm into 3 equal parts, called

* Rhenish wine is more frequently imported in bottles than in Aums.

fluid scruples; and the fluid scruple into 20 equal parts, called minims.]

20 Minims (m.)	=	1 fluid Scruple (fl. ℈)
3 fluid Scruples	=	1 " Drachm (fl ʒ.)
8 " Drachms	=	1 " Ounce (fl. ℥.)
20 " Ounces	=	1 Pint (O.)
8 Pints	=	1 Imperial GALLON (C.)

C, the symbol for "gallon," is the initial letter of the Latin word *congius*—the name of an ancient vessel which contained a little more than a gallon. O, the symbol for "pint," is the initial of the Latin word *octarius*. Minim is a contraction of *minimum*—a name naturally given to the *smallest* measure. A minim of distilled water weighs ·9 grains; and a fluid ounce of distilled water, therefore, weighs 432 grains—a little less than an Avoirdupois ounce (437½ grains). Very small quantities of liquid medicines are frequently expressed in "drops," instead of in minims; and although a drop is a variable quantity—the size depending partly upon the nature of the liquid, and partly also upon the shape of the mouth of the vessel, 3 drops are regarded as representing, in most cases, 2 minims.*

WEIGHT.

There are two standards of weight: the Avoirdupois pound of 7,000 grains, and the Troy pound of 5,760 grains. The articles sold by Troy weight are—gold, silver, platinum, and precious stones. All other articles are sold by Avoirdupois weight. The Troy pound was made a legal standard in the reign of Henry VII., and the Avoirdupois pound in the reign of Elizabeth. A "grain" originally meant the weight of a *grain of wheat*, "taken from the middle of the ear."

Avoirdupois Weight.

16 Drams (drs.)	=	1 Ounce (oz.)
16 Ounces	=	1 POUND (lb.), of 7,000 grs.
28 Pounds	=	1 Quarter [of a Hundred-weight] (qr.)
4 Quarters	=	1 Hundred-weight (cwt.)
20 Hundred-weights	=	1 Ton.

The Hundred-weight is also divided into 8 stones of 14 pounds each.

In every-day life, the ounce is divided (not into "drams," but) into *half*-ounces and *quarter*-ounces.

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* There are some liquids of which a drop is more than a minim. A drop of Prussic acid, for example, is as much as a minim and a half.

The following Avoirdupois weights are employed in the Wool trade, and are usually classed under the head of

Wool Weight.

7	Pounds	=	1 Clove
2	Cloves	=	1 Stone
2	Stones	=	1 Tod
6½	Tods	=	1 Wey
2	Weys	=	1 Sack
12	Sacks	=	1 Last

Although wool-*growers* sell their wool by the ordinary stone of 14 pounds, wool-*staplers*, in their dealings with each other and with the general public, employ a stone of 15 lbs.*

Miscellaneous Avoirdupois Weights.

		lbs.
A Stone of—		
Hemp	=	32
Glass	=	5
Butchers' Meat	=	8
Iron wire, up to "20 gauge"	=	10½
Iron wire, above "20 gauge"	=	10

		lbs.
A Bundle of—		
Iron	=	56
Iron wire, up to "20 gauge"	=	63
Iron wire, above "20 guage"	=	60

		lbs.
A Firkin of—		
Butter	=	56
Soap	=	64

		cwt.
A Sack of—		
Coals	=	2
Flour	=	2½

		lbs.
A Barrel of—		
Soap	=	256
Flour	=	196
Gunpowder	=	112
Anchovies	=	30

		lbs
A Gallon of—		
Oil	=	9
Salt	=	7

		lbs.
A Truss of—		
Straw	=	36
Old Hay	=	56
New Hay	=	60

A Load (36 Trusses) of—		cwt.	qrs.	lbs.
Straw	=	11	2	8
Old Hay	=	18	0	0
New Hay	=	19	1	4

A Fother of Lead = 19½ cwt.; a Pocket of Hops = 1 cwt.; a Quintal of Fish = 1 cwt.; a Bag of Rice = 168 lbs.; a Chest of Tea = 84 lbs.; a Clove of Cheese = 8 lbs.

* Quantities of wool are sometimes expressed in "Packs." A wool-grower's Pack is 3¼ cwt. (364 lbs.); a wool-stapler's Pack, 2¼ cwt. (240 lbs.)

Apothecaries' Weights.

[In the compounding of medicines, apothecaries divide the Avoirdupois ounce into 8 equal parts, called drachms, of 54·69 grains each, and the drachm into 3 equal parts, called scruples, of 18·23* grains each.]

18·23* Grains	=	1 Scruple (℈)
3 Scruples	=	1 Drachm (ʒ)
8 Drachms	=	1 Avoirdupois ounce (℥)

For a considerable time previously to the year 1864, these weights were used only by the apothecaries of Ireland—the British Apothecaries employing, under the same names, corresponding sub-divisions of the *Troy* ounce ("20 grains = 1 scruple; 3 scruples, or 60 grains = 1 drachm; 8 drachms, or 480 grains = 1 Troy ounce"). So that a British scruple exceeded an Irish scruple by nearly 2 grains, whilst a British drachm exceeded an Irish drachm by more than 5 grains. The subdivisions, given above, of the Avoirdupois ounce are now employed, in the compounding of medicines, all over the United Kingdom. In the buying of their medicines, however, apothecaries employ the ordinary standards of Avoirdupois weight.

Troy Weight.

24 Grains (grs.)	=	1 Pennyweight (dwt.)
20 Pennyweights	=	1 Ounce (oz.)
12 Ounces	=	1 POUND (℔), of 5,760 grs.

"Pennyweight" takes its name from the ancient English penny—a silver coin about the size of the present three-penny piece, and employed not only as a coin, but also as a weight; just as the bronze money of the present day might be so employed. This penny was the first silver coin issued in England, and for a long time it was the only one in circulation. Up to the time of Edward I. it was deeply indented with a cross; so that it could easily be broken into halves (called "half-pence") and quarters (called "fourthings" or *farthings*). In this king's reign the cross was dispensed with, having been rendered unnecessary by a supply of *round* half-pence and farthings ("fourthings")—then issued for the first time; and it was decreed that the silver penny should be equal in weight to 32 grains of wheat ("taken from the middle of the ear"), and should be recognised as a standard of weight—the "*penny*-weight." It was also decreed that 20 such pennyweights should constitute an "ounce." The pennyweight was afterwards reduced to 24 grains.

* More accurately, 18·22917 grains.

Diamond Weight.

16 Parts = 1 Grain
4 Grains = 1 Carat

The "grain" employed by lapidaries—in the weighing of diamonds, rubies, and emeralds—is a special weight, equal to four-fifths of an ordinary grain. So that the lapidaries' "carat" is only $3\frac{1}{5}$ ordinary grains.

"Carat" is employed by goldsmiths as a name for the *twenty-fourth* part of a piece of gold whose fineness they want to express. Thus, gold is said to be "20 carats fine" when 20 parts out of 24 are pure gold; "18 carats fine," when 18 parts out of 24 are pure gold; &c. A sovereign is "22 carats fine"—22 twenty-fourths of it being pure gold.

Division of the Circle—Angular Measure.

The term *degree* is applied to the 360th part of the circumference of a circle, and also to the 90th part of a right angle. The 60th part of a degree is called a *minute*; and the 60th part of a minute, a *second*. Degrees, minutes, and seconds are marked $^\circ$, $'$, and $''$, respectively. Thus, 7 degrees 8 minutes 9 seconds would be written $7^\circ\ 8'\ 9''$:

Division of the Circle.	*Angular Measure.*
60 Secs. = 1 Min.	60 Secs. = 1 Min.
60 Mins. = 1 Deg.	60 Mins. = 1 Deg.
360 Degs. = 1 Circum.	90 Degs. = 1 Right Angle

The second was formerly divided into 60 equal parts, called *thirds*; and the third into 60 equal parts, called *fourths*. Thirds and fourths are now expressed as a decimal of a second; and minutes and seconds are often expressed as a decimal of a degree. It is hardly necessary to observe that whilst a degree of the circumference of a circle—an "arcual" degree, as it is sometimes termed—is long or short according to the size of the circle, an "angular" degree is always the same, right angles being all equal.

A Dozen of articles = 12	A Quire of Paper = 24 sheets
,, Score ,, ,, = 20	,, Ream ,, ,, = 20 quires
,, Gross ,, ,, = 144	

Time.

60 Seconds	= 1 Minute
60 Minutes	= 1 Hour
24 Hours	= 1 Day
7 Days	= 1 Week
52 Weeks and 1 Day, or 365 Days	= 1 "Common" Year
52 Weeks and 2 Days, or 366 Days	= 1 "Leap" Year

The seven DAYS OF THE WEEK: Sunday, Monday, Tuesday, Wednesday, Thursday, Friday, Saturday.*

The twelve CALENDAR MONTHS: January, February, March, April, May, June, July, August, September, October, November, December.

The four SEASONS OF THE YEAR:

Spring—	embracing	February, March, and April;
Summer—	„	May, June, and July;
Autumn—	„	August, September, and October;
Winter—	„	November, December, and January.

Number of DAYS IN EACH MONTH:

		days.			days.
January	=	31	July	=	31
February	=	28 in a com. yr. / 29 „ leap yr.	August	=	31
			September	=	30
March	=	31	October	=	31
April	=	30	November	=	30
May	=	31	December	=	31
June	=	30			

"Calendar" Months are so called in contradistinction to "lunar" months, which consist of 28 days each, and 13 of which represent a year—or rather 364 days. "Month" originally meant the time (28 days) in which the moon was supposed to complete a revolution, and this is the sense in which the expression "*lunar* month" is now used, although it has

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* The days of the week are named after the seven "planets" known to the early Egyptian astronomers, and amongst which were included the sun and the moon. Subjoined are the names of those "planets," and also the Latin names (still used in Parliamentary and in many legal documents) of the days of the week:—

Saturn . . .	*Dies Saturni*	(Saturn's Day)
Sun	*Dies Solis*	(Sun's Day)
Moon . . .	*Dies Lunae*	(Moon's Day)
Mars . . .	*Dies Martis*	(Mar's Day)
Mercury . .	*Dies Mercurii*	(Mercury's Day)
Jupiter . . .	*Dies Jovis*	(Jupiter's Day)
Venus . . .	*Dies Veneris*	(Venus's Day)

The English names Saturday, Sunday, Monday are derived directly from Saturn, Sun, and Moon, respectively. Instead, however, of "Marsday," "Mercuryday," "Jupiterday," and "Venusday," we have Tuesday (Tuesco's day), Wednesday (Woden's day), Thursday (Thor's day), and Friday (Frigga's day): derived from Tuesco, Woden, Thor, and Frigga, respectively—the Mars, Mercury, Jupiter, and Venus of Scandinavian and Anglo-Saxon mythology.

been ascertained that the period of the moon's revolution is only 27·32166 days (27 dys. 7 hrs. 43 min. 11½ sec.). "Calendar" is derived from *Calends*, a name given by the Romans to the first day of a month, and afterwards extended to the months themselves.

The following rhyme respecting the lengths of the months deserves to be quoted, if only on account of its antiquity:—

> "Thirty days hath November,
> April, June, and September;
> February twenty-eight alone,
> And all the others thirty-one:
> But leap-year, coming once in four,
> Gives February one day more."

Of the many rules formerly given for the remembering of the "long" and the "short" months, perhaps the most curious and ingenious is this: Close the hand, and (disregarding the thumb) tell over the months in regular succession upon the four knuckles and the three hollows between the knuckles; passing—not from knuckle to knuckle directly, but—from knuckle to hollow, and from hollow to knuckle, and returning to the first knuckle after getting to the fourth. All the "long" months will then be found *on*, and all the "short" ones *between*, the knuckles.

It is worthy of remark that whilst other standards (such as those of value, length, &c.)—being perfectly arbitrary—are different in different countries, the standards of Time are everywhere the same, because fixed by Nature herself. In all countries, the *day* is the principal standard of Time, and naturally so; the commencement and the termination of the day being marked by phenomena which are observed by everybody. No such phenomena, however, indicate the precise moment at which a *year* begins or ends; and in former times, when astronomy and the kindred sciences were very imperfectly understood, the estimates formed of the length of the year were little better than mere conjectures.

The most accurate estimate was that of the Egyptians, whose year consisted of 360* days—12 months of 30 days each. The Greek year consisted of only 354 days, and was divided into 12 months of 29 and 30 days each, alternately. The Roman year, as established by Romulus, consisted of only 304 days, and was divided into 10 months, of which six contained 30, and the remaining four 31 days each. The names of those 10 months were: *Mars*, *Aprilis*, *Maia*, *Junius*, *Quintilis* (fifth), *Sextilis*

* The length of the Egyptian year is said to have given rise to the division of the circle into 360 degrees.

(sixth), *September* (seventh), *October* (eighth), *November* (ninth), *December* (tenth). The first month took its name from Mars, who, according to an old Roman fable, was the father of Romulus; the name of the second month is said to have been derived from the Latin word *aperire* (to open)—suggestive of the change which vegetable life undergoes in spring; the third month took its name from Maia, the mother of Mercury; and the fourth from Juno, the queen of the gods. The names of the remaining six months explain themselves.

The year of 304 days was soon found to be too short; and in the reign of Numa, second King of Rome, two additional months were introduced—January and February: the former, which became the first month, was named after Janus, a Roman deity, who was supposed to preside over the *beginning* of every important enterprise; the latter month took its name from Februus, another Roman deity, who was supposed to preside over the dead, and whose rites were celebrated about that time of the year. The months originally known as Quintilis and Sextilis were afterwards called Julius (*July*) and Augustus (*August*), respectively, in compliment to two of the Roman emperors.

Of the 12 months into which (at first) Numa's year was divided, four contained 31 days each, seven others 29 days each, and the remaining month (February) 28 days. The experience of agriculturists and others proving that even this year (355 days) did not embrace the four seasons, Numa decreed that a thirteenth month, called Mercedonius, should be introduced into every second year, and should consist of 22 and 23 days, alternately. After Numa's time, however, great irregularities occasionally occurred in the carrying out of this arrangement—chiefly on account of the liberties taken with the extra or "intercalary" month, which was sometimes shortened, sometimes lengthened, and sometimes omitted altogether. The errors occasioned by these irregularities went on accumulating, year after year, until the time of Julius Cæsar, when they were found to have amounted to more than two months—when a day which properly belonged to the latter end of September, for example, was set down as belonging to the beginning of December.

Having, with the aid of Sosigenes, a distinguished Egyptian astronomer, ascertained the length of the year to be—as nearly as it was possible for the science and the astronomical instruments of that day to ascertain it—365¼ days, Cæsar caused the year in which his reign began (the year 708 from the founding of Rome, and 45 B.C.: the "Year of Confusion," as it afterwards came to be called) to consist of 445 days, of which 355 represented the twelve ordinary months,

23 the extra month Mercedonius, and the remaining 67 two "extraordinary" months (of 34 and 33 days)—introduced between November and December. Cæsar next ordained that, thenceforward, three successive years should consist of 365 days each, but that—as a compensation for the omission of the quarter-day—every fourth year should contain 366 days, and that the additional day (representing the 4 quarter-days) should be given to February. It was further ordained that the 12 months (Mercedonius was abolished) should be long and short, alternately, and that the long months should consist of 31, and the short months of 30 days each—with the exception of February, which, it was decreed, should contain 29 days in a common, and 30 days in a leap year.

This arrangement was afterwards altered at the instance of the Emperor Augustus, whose jealousy was excited by the circumstance that *his* month (Augustus—previously known as Sextilis) was shorter than Cæsar's (July—previously known as Quintilis); and the Romans, to gratify the vanity of Augustus, gave August an additional day at the expense of February. They also gave October and December an additional day each, at the expense of September and November, in order that—so far as the change in the length of August rendered the arrangement practicable—the months should still be long and short, alternately.

The exact length of the year being 365d. 5h. 48m. 49sec., the "Julian" year of 365d. 6h. was a little more than 11 minutes too long; and this error, although too small to have been perceptible during the lifetime of any one individual or generation, had amounted to a *day* in 129 years, to a *week* in 903 years, and to *ten days* in the year A.D. 1582, when Pope Gregory XIII. undertook the reform of the calendar.* At this Pontiff's suggestion, the day which, in 1582, would be the 5th October according to the Julian calendar, was called the 15th October; and it was further arranged that, as an error of 11 minutes in one year represented an error of 3 days in 400 years, a day should, thenceforward, be omitted from each of three centuries out of every four. According to the Julian calendar, the closing year of *every* century would be a leap year, but Pope Gregory ordained that the closing year of only every *fourth* century

* At the end of 23,558 years the error, if allowed to remain so long uncorrected, would have amounted to *six months*, in which case the longest day would occur in December, and the shortest day in June; the seed-time would be in Autumn, and the harvest in Spring; and people would be wearing their warmest clothes in Summer, and their lightest in Winter.

should be so regarded.* The arrangement will be understood from the subjoined details:—

1700, 1800, 1900	Common years.	2500, 2600, 2700	Common years.
2000	Leap year.	2800	Leap year.
2100, 2200, 2300	Common years.	2900, 3000, 3100	Common years.
2400	Leap year.	3200	Leap year.

The Gregorian Calendar—or "New Style," as it is called—has been introduced into every civilized country except Russia, where time is still computed according to the Julian calendar, or "Old Style." The two styles now differ in their reckoning by 12 days. The New Style was not introduced into the United Kingdom until the year 1752, when the error had amounted to 11 days (the year 1700 having been reckoned a leap year), and when, upon the authority of an Act of Parliament, the 3rd September (Old Style) was called the 14th September.† For some time after the change of style, it was not unusual for lawyers and others to express an important date according to *both* styles, in the form of a fraction; thus:—

$$\frac{\text{27th March,}}{\text{7th April,}}\text{1753.}$$

The year did not always begin on the 1st January. The French year, for example, began on Christmas Day in the time of Charlemagne, and on Easter Sunday in the time of the Capet monarchs. In the United Kingdom, previously to the change of style, the year began on the 25th March ("Lady Day")—a circumstance which explains the practice that still prevails of dating leases from that day instead of from the 1st January, and of paying half-yearly gales of rent on Lady Day and Michaelmas Day (29th September), instead of in June and December.

* With these exceptions, a leap year is known by its leaving no remainder when divided by 4. Thus, 1796 was, but 1839 was not, a leap year.

† To many persons still living, the expression "As long as the 11th of June" (Old Style) is quite familiar.

The length of a pendulum which, in a vacuum, would, at the level of the sea, vibrate seconds in the latitude of London, has been ascertained to be 39·1393 inches. This connexion between time and length affords a means by which the existing standards, if destroyed, could easily be replaced. Thus, the seconds pendulum would determine the Yard—the length of the former being to that of the latter as 39·1393 to 36. The Yard being determined, both the Square Yard and the Cubic Yard would also be determined. The Yard would likewise determine an inch, which would determine a cubic inch, which, in its turn, would determine the Gallon (277·274 cubic inches). The Gallon (which holds 10 lbs. Avoir. of distilled water) would determine the Avoir. Pound, and therefore also the Troy Pound. And a grain—obtainable from either of the two Pounds—would determine the Sovereign, which weighs 5 dwts. 3¼ grs:

Standard of	Value,	—	the	Pound or	Sovereign	
	Length,	—	,,		Yard	
	Surface,	—	,,	Square	Yard	
	Solidity,	—	,,	Cubic	Yard	
	Capacity,	—	,,		Gallon	
	Weight,	—	,,	Avoir.	Pound	(See p. 76)
			,,	Troy	Pound	
	Arcual measure, Angular ,,		,,		Degree	(See p. 73)
	Time,	—	,,		Day	

Besides the standards just mentioned, several local ones—retained under the influence of usage and prejudice, although not recognised by law—continue to be employed in different parts of the United Kingdom; and, what is still more perplexing, a local and a legal standard have the same name in some cases, so that the one is liable to be confounded with the other. In Ireland, up to the close of 1862, the Registrar-General used to express quantities of wheat, barley, and oats, in *barrels* of 20 st., 16 st., and 14 st. each, respectively—these being the three meanings attached to the word "barrel" in most parts of the country. In some places, however, "barrel" had other meanings. Thus, wheat was sold in Dublin by the barrel of 282 lbs. (20¼ st.), whilst barley and oats were sold in Sligo by the barrel of 24 stone. Grain was sold in Belfast by the hundred-weight, and in Limerick by the stone. The hundred-weight had two meanings in Belfast—112 lbs. in the case of corn, and 120 lbs. in the case of pork. Moreover, the Belfast people sold flax by the stone of 16¾ lbs. in one part of the town, and by the stone of 24½ lbs. in another part. A stone of flax meant 16 lbs. in Dublin, and 24 lbs. in Downpatrick.

All these anomalies, and others into the details of which it is now unnecessary to enter, either have disappeared or are gradually disappearing under the operation of an Act of Parliament which took effect in the beginning of 1863. It is difficult to understand why this Act was not extended to Great Britain, where there are upwards of 50 different standards (including 18 varieties of bushel) for the measurement of grain; also, 11 different standards of length, and 16 of surface (for the measurement of land); and where

Brickwork is measured by the perch of $22\frac{1}{2}$ cubic feet, the perch of 36 cubic feet, and the rod of $272\frac{1}{4}$ square feet × $1\frac{1}{2}$ bricks thick;

Potatoes are sold by the measure of 84 lbs., the long measure of 90 lbs., the winch (Winchester bushel*), the load of 240 lbs., the sack of 10 pecks, the sack of 3 bushels, the bag of 140 lbs., and the hundred-weight of 120 lbs;

Butter is sold by the pound of 18 oz., the pound of 24 oz., the dish of 24 oz., the dish of 22 oz., the pint of 20 oz., and the roll of 24 oz.;

Flour is sold by the aigendale of 8 lbs., the gallon of 8 lbs. 1 oz., the pack of 240 lbs., and the sack of 5 bushels; and

Coals are sold by the long hundred-weight of 120 lbs., the peck of 1,209 cubic inches, the corf of 1 cwt., the corf of 3 cwt., and the corf of "3 to 4" tons.

Grain is sold in Glasgow by the boll, of which there are four varieties—a boll of wheat being 240 lbs., a boll of barley 320 lbs., a boll of oats 264 lbs., and a boll of Indian corn 280 lbs. In Liverpool, wheat is sold by the bushel of 70 lbs., barley by the bushel of 60 lbs., and oats by the bushel of 45 lbs.† Wheat is sold in Bridgend (Wales) by the bushel of 168 lbs., and in Pwlheli (also in Wales) by the bushel of 252 lbs. In Saltash (Cornwall), wheat is sold by the bushel of 8 gallons, and oats by the bushel of 24 gallons. In Manchester, a bushel of English wheat means 60 lbs., and of American wheat 70 lbs. In Preston, barley is sold by the barrel, which means 224 lbs. or 240 lbs., according as the barley is "for malting" or "for grinding." Grain is likewise sold by the "bag" (7 varieties), by the "load" (6 varieties), by the "hobbet" (5 varieties), by the "weight" (3 varieties), by the "measure" (2 varieties), by the

* In 1697 the capacity of the Winchester bushel was fixed at 2150·42 cubic inches—a capacity very slightly in excess of that of $7\frac{3}{4}$ Imperial gallons.

† A new standard, called the *centner*, representing a weight of 100 lbs., was lately introduced into the Liverpool corn-market.

"stack," by the "coomb," by the "windle," by the "strike," &c.

It is difficult to reconcile this multiplicity of weights and measures with the fact that a uniform set of standards existed in England before the Conquest, or with the decree of Richard I.—a decree subsequently confirmed by Magna Charta—that there should be but "one weight and one measure throughout the realm." Since the middle of the last century, the weights and measures of the United Kingdom have engaged the attention of no fewer than eight Parliamentary Committees, as well as of one or two Royal Commissions, and have been the subject of several legislative enactments. The labours of the last Committee, that sat in 1862, have resulted in the legalization, by Parliament, of what is called the Metric System, a detailed account of which will be found farther on.

REDUCTION.

70. The conversion of concrete numbers into others of higher or lower denominations (but of the *same kind*) is termed REDUCTION.

71. Reduction divides itself into two parts: (1.) *Descending* Reduction, or the reduction of numbers to lower denominations; and (2.) *Ascending* Reduction, or the reduction of numbers to higher denominations.

Under the head of Descending Reduction would come the reduction of pounds to farthings, of miles to yards, of acres to perches, &c.; whilst under the head of Ascending Reduction would come the reduction of farthings to pounds, of yards to miles, of perches to acres, &c.

72. An exercise in Descending Reduction is merely a particular application of Simple Multiplication, or of Simple Multiplication *and* Simple Addition—according as the number to be reduced is simple or compound.* An exercise in Ascending Reduction is a particular application of Simple Division.*

* In the reduction of perches to yards, or of yards to perches, both multiplication and division have to be employed.

Descending Reduction.

Example I.—How many farthings are there in 47 pounds?

We first reduce the pounds to shillings, by multiplying by 20; next, the shillings to pence, by multiplying by 12; and then the pence to farthings, by multiplying by 4. The number of shillings in one pound being 20, the number in 47 pounds must be 47 times 20, or 20 times 47—that is, 940. The number of pence in one shilling being 12, the number in 940 shillings must be 940 times 12, or 12 times 940—that is, 11,280. And, the number of farthings in one penny being 4, the number in 11,280 pence must be 11,280 times 4, or 4 times 11,280—that is, 45,120. So that 47 pounds are equal in amount to 940 shillings, or to 11,280 pence, or to 45,120 farthings.

```
  £47
   20
 ----
 940s.
   12
 ----
11280d.
    4
 -----
45120f.
```

Example II.—Reduce £28 14*s*. $5\frac{1}{2}$*d*. to farthings.

In 28 pounds there are (28×20=)560 shillings; therefore, in 28 pounds *and* 14 shillings there are (560+14=)574 shillings. In 574 shillings there are (574×12=)6,888 pence; therefore, in 574 shillings *and* 5 pence —or in £28 14*s*. 5*d*.—there are (6,888+5=) 6,893 pence. In 6,893 pence there are (6,893×4=)27,572 farthings; therefore, in 6,893 pence *and* a halfpenny [2 farthings]— or in £28 14*s*. $5\frac{1}{2}$*d*.—there are (27,572+2=) 27,574 farthings. The work is shown in the margin. Instead of first setting down the product of 28 by 20, and afterwards adding 14, we as it were combine the two operations—writing 4, the units' figure of the given number of shillings, instead of the units' figure (0) of the product, and adding 1, the tens' figure of the shillings, to the tens' figure (6) of the product. In like manner, when multiplying 574 by 12 we add 5 to the product: "12 fours = 48; [48] and 5 = 53;" &c. And when multiplying 6,893 by 4, we add 2 (the number of farthings in the halfpenny) to the product: "4 threes = 12; [12] and 2 = 14;" &c.

```
  £   s.  d.
  28  14  5½
  20
 ---
 574s.
  12
 ----
6893d.
   4
 -----
27574f.
```

EXAMPLE III.—Reduce 23 miles to yards.

We first reduce the miles to furlongs, by multiplying by 8; next, the furlongs to perches, by multiplying by 40; and then the perches to yards, by multiplying by $5\frac{1}{2}$. The number of furlongs in one mile being 8, the number in 23 miles must be 23 times 8, or 8 times 23—that is, 184. The number of perches in one furlong being 40, the number in 184 furlongs must be 184 times 40, or 40 times 184—that is, 7,360. And, the number of yards in one perch being $5\frac{1}{2}$, the number in 7,360 perches must be 7,360 times $5\frac{1}{2}$, or $5\frac{1}{2}$ times 7,360—that is, 40,480: 5 times 7,360=36,800; one-half of 7,360(7,360÷2=)3,680; $5\frac{1}{2}$ times 7,360=(368.800 +3,680=)40,480. So that 23 miles are equal in length to 184 furlongs, or to 7,360 perches, or to 40,480 yards.

23 m.
8
184 fur.
40
7360 per.
$5\frac{1}{2}$
36800
3680
40480 yds.

EXAMPLE IV.—Reduce 17 m. 6 fur. 28 per. 4 yds. to yards.

In 17 miles there are (17×8=) 136 furlongs; therefore, in 17 miles *and* 6 furlongs there are (136+6=) 142 furlongs. In 142 furlongs there are (142×40=) 5,680 perches; therefore, in 142 furlongs *and* 28 perches—or in 17 m. 6 fur. 28 per.—there are (5,680+28=) 5,708 perches. In 5,708 perches there are (5,708×$5\frac{1}{2}$=) 31,394 yards; therefore, in 5,708 perches *and* 4 yards—or in 17 m. 6 fur. 28 per. 4 yds.—there are (31,394+4=) 31,398 yards. In practice, we add the 4 yds. to 5 times 5,708, and then add the half of 5,708 to the result.

m.	fur.	per.	yds.
17	6	28	4
8			
142 fur.			
40			
5708 per.			
$5\frac{1}{2}$			
28544			
2854			
31398 yds.			

73. Rule for Descending Reduction: Reduce the highest denomination to the next lower, by multiplying by the number indicating how many units of the lower denomination are contained in one of the highest; and to the product add as much of the given number as belongs to this lower denomination. Treat the result in the same way, and continue the process until the required denomination is obtained.

ASCENDING REDUCTION.

EXAMPLE I.—Reduce 45,120 farthings to pounds.

We first reduce the farthings to pence, by dividing by 4; next, the pence to shillings, by dividing by 12; and then the shillings to pounds, by dividing by 20. If the farthings were formed into penny packages, each package would contain 4, and the total number of such packages would be (45,120÷4=)11,280; so that 45,120 farthings are worth 11,280 pence. Again: if the pence were formed into shilling packages, each package would contain 12, and the total number of such packages would be (11,280÷12=) 940; so that 11,280 pence are worth 940 shillings. In like manner, if the shillings were formed into pound packages, each package would contain 20, and the total number of such packages would be (940÷20=)47; so that 940 shillings are worth 47 pounds. We thus find that 45,120 farthings, 11,280 pence, 940 shillings, and 47 pounds are all equal in amount.

4)45120*f.*
12)11280*d.*
2|0)94|0*s.*
£47

EXAMPLE II.—Reduce 27,574 farthings to pounds.

Dividing by 4, we find that in 27,574 farthings there are 6,893 pence, and two farthings or one half-penny over. Dividing by 12, we next find that in 6,893 pence there are 574 shillings, and 5 pence over. Lastly, dividing by 20, we find that in 574 shillings there are 28 pounds, and 14 shillings over. So that 27,574 farthings are equal in amount to 6,893½ pence, or to 574 shillings and 5½ pence, or to £28 14*s.* 5½*d.*

4)27574*f.*
12)6893½*d.*
2|0)57|4*s.* 5½*d.*
£28 14*s.* 5½*d.*

NOTE.—We reduce shillings to pounds in this way: we first convert the shillings into half-sovereigns, by dividing by 10; and then the half-sovereigns into pounds, by dividing by 2. In dividing by 10, we simply "cut off," for remainder, the most right-hand figure of the dividend, and take the remaining figure or figures for quotient. Thus, 940 shillings [Ex. I.] are equal in amount to (940÷10=) 94 half-sovereigns, which are equivalent to (94÷2=) 47 pounds. In like manner, 574 shillings [Ex. II.] are equal in amount to (574÷10=) 57 half-sovereigns and 4 shillings, or to £28 14*s.*,—57 half-sovereigns being equivalent to (57÷2=) 28 pounds and 1 half-sovereign, or to £28 10*s.*

EXAMPLE III.—Reduce 40,480 yards to miles.

We first reduce the yards to perches, by diving by $5\frac{1}{2}$,—or by multiplying by 2 and dividing by 11; next, the perches to furlongs, by dividing by 40; and then the furlongs to miles, by dividing by 8. In 40,480 yards there are $(40,480 \times 2 =)$ 80,960 *half*-yards, and in a perch there are $(5\frac{1}{2} \times 2 =)$ 11 *half*-yards. Dividing 80,960 by 11, therefore, we find that in 80,960 half-yards, or 40,480 yards, there are 7,360 lengths of 11 half-yards or $5\frac{1}{2}$ yards each—that is, 7,360 perches. Dividing 7,360 by 40, we next find that in 7,360 perches there are 184 lengths of 40 perches each—that is, 184 furlongs. Lastly, dividing 184 by 8, we find that in 184 furlongs there are 23 lengths of 8 furlongs each—that is, 23 miles. So that 40,480 yards are equal in length to 7,360 perches, or to 184 furlongs, or to 23 miles.

```
      40480 yds.
          2
      -----
  11)80960 half-yds.
      -----
  40)7360 per.
      ----
   8)184 fur.
      ---
      23 m.
```

EXAMPLE IV.—Reduce 31,398 yards to miles.

Reducing the yards to half-yards, and dividing by 11 (the number of half-yards in a perch), we obtain 5,708 for dividend, and 8 for remainder—that is, 5,708 perches and 8 half-yards. Instead, however, of the 8 half-yards, we set down $(8 \div 2 =)$ 4 yards. Dividing by 40, we next find that in 5,708 perches there are 142 furlongs and 28 perches. Lastly, dividing by 8, we find that in 142 furlongs there are 17 miles and 6 furlongs. So that 31,398 yards are equal in length to 17 m. 6 fur. 28 per. 4 yds.

```
      31398 yds.
          2
      -----
  11)62796 half-yds.
      -----
  40)5708 per. 4 yds.
      ----
   8)142f. 28p. 4yds.
      ---
     17m. 6f. 28p. 4y.
```

NOTE.—In dividing by 40, we employ the factors 10 and 4. Thus, dividing 5,708 [Ex. IV.] by 10, we obtain 570 for quotient, and 8 for remainder. Next, dividing 570 by 4, we obtain 142 for quotient, and 2 for remainder. This second quotient we set down as the "true" one. To obtain the "true" remainder, we multiply 10, the first divisor, by 2, the second remainder, and add 8, the first remainder, to the product: $10 \times 2 = 20$; $20 + 8 = 28$.

74. Rule for Ascending Reduction: First, reduce the given number to the next higher denomination, by dividing by the number which indicates how many units of the given denomination are contained

in one of the next higher; then, reduce this new denomination—if it be not the required one—to the next higher, and continue the process until the required denomination is obtained.

THE COMPOUND RULES.

COMPOUND ADDITION.

75. Addition is called COMPOUND when the addends are compound numbers (of the *same kind*).

76. Compound addition is performed by means of Simple Addition and Ascending Reduction.

EXAMPLE I.—A merchant received £16 12*s*. 7¾*d*. on Monday, £23 17*s*. 8½*d*. on Tuesday, £39 18*s*. 4¼*d*. on Wednesday, £65 11*s*. 3½*d*. on Thursday, £46 19*s*. 5¾*d*. on Friday, and £50 16*s*. 2½*d*. on Saturday: how much did he receive during the week?

Arranging the addends as in the margin, we find, by means of Simple Addition, that, altogether, the merchant received 239 pounds, 93 shillings, 29 pence, and 13 farthings (each halfpenny being reckoned 2 farthings). In practice, however, a sum of money is not written under this form—as much as possible of the amount being expressed in pounds, as much as possible of the remainder in shillings, and as much as possible of what then remains in pence: so that the number of shillings can never exceed 19, nor the number of pence 11, nor the number of farthings 3.

£	*s*.	*d*.
16	12	7¾
23	17	8½
39	18	4¼
65	11	3½
46	19	5¾
50	16	2½
239	93	$29\frac{13}{4}$

We therefore reduce the farthings to pence, which we "carry" to the pence column; we next reduce the pence to shillings, which we carry to the shillings column; and we then reduce the shillings to pounds, which we carry to the pounds column. The required sum is thus found to be £243 15*s*. 8¼*d*. The work proceeds in this way:—$2+3+2+1+2+3=13$; 13 farthings $=$ 3¼*d*.; [set down] ¼*d*., and carry 3;—3 (the carried figure) $+2+5+3+4+8+7=32$; 32 pence $=$ 2*s*. 8*d*.; [set down] 8*d*., and carry 2; &c.

£	*s*.	*d*.
16	12	7¾
23	17	8½
39	18	4¼
65	11	3½
46	19	5¾
50	16	2½
243	15	8¼

EXAMPLE II.—A farmer has four farms, containing, respectively, 73A. 2R. 30P., 65A. 3R. 24P., 49A. 1R. 17P., and 36A. 2R. 35P.: how much land does he hold altogether?

Arranging the addends as in the margin, we first find that the total number of perches is (35+17+24+30=) 106. Dividing by 40, the number of perches in a rood, we convert 106 perches into 2 roods and 26 perches. Setting down the 26 perches, and carrying the 2 roods, we next find that the total number of roods is (2+2+1+3+2=) 10. Dividing by 4, the number of roods in an acre, we convert 10 roods into 2 acres and 2 roods. Setting down the 2 roods, and carrying the 2 acres, we finish by finding the total number of acres = 225. So that the required sum is 225A. 2R. 26P.

A.	R.	P.
73	2	30
65	3	24
49	1	17
36	2	35
225	2	26

77. Rule for Compound Addition: Set down the addends—one below the other—in such a way that all the numbers of the same denomination shall stand in the same vertical column. Draw a horizontal line to separate the addends from their sum. Then find, by Simple Addition, the total number of units of the lowest denomination, and divide by the number indicating how many such units are contained in a unit of the next higher denomination; write the remainder, if there be one, at the bottom of the most right-hand column, and add (or "carry") the quotient to the column of the next higher denomination. Proceed in the same way with each of the other columns, successively.

COMPOUND SUBTRACTION.

78. Subtraction is called COMPOUND when the two numbers whose difference is required (and which must be of the *same kind*) are compound numbers; or when only one of the numbers is compound; or, again, when both numbers are simple, but of different denominations.

"What is the difference between 5*s.* and 3*s.* 4*d.*?" "By how much does £10 exceed 17*s.*?" Both these questions belong to Compound Subtraction.

79. Compound Subtraction is performed by means of Simple Subtraction* and Descending Reduction.

EXAMPLE I.—Out of a purse containing £23 11*s.* 9¼*d.* a sum of £4 17*s.* 6¾*d.* was paid: how much remained?

Let us suppose the £4 17*s.* 6¾*d.* to have been paid in four instalments of ¾*d.*, 6*d.*, 17*s.*, and £4, respectively. The number of farthings in the purse being only 1, the purse-bearer must have paid the first instalment (¾*d.*) by giving a penny, and getting back (1*d.* – ¾*d.* =) ¼*d.* After payment of the first instalment, therefore, the purse contained 1*d.* less, and ¼*d.* more than at first; so that the number of farthings then was (1 + 1 =) 2. After payment of the second instalment (6*d.*), the number of pence in the purse was diminished by (1+6=) 7; so that the number of pence remaining in the purse was (9 – 7=) 2. The number of shillings in the purse being only 11, the purse-bearer must have paid the third instalment (17*s.*) by giving a pound, and getting back (£1–17*s.*=) 3*s.* After payment of the third instalment, therefore, the purse contained £1 less, and 3*s.* more than at first; so that the number of shillings then was (11+3=) 14. After payment of the last instalment (£4), the number of pounds was diminished by (1+4=) 5; so that the number of pounds remaining in the purse was (23–5=) 18. The required remainder is thus found to be £18 14*s.* 2½*d.*

£	s.	d.
23	11	9¼
4	17	6¾
18	14	2½

EXAMPLE II.—A butcher had in his stall 10 cwt. 2 qrs. 18 lbs. of beef, and of this quantity he sold 7 cwt. 3 qrs. 13 lbs.: how much had he remaining?

Let us suppose the 10 cwt. 2 qrs. 18 lbs. to have been cut up into 10 pieces of 1 cwt. each, 2 pieces of 1 qr. each, and 18 pieces of 1 lb. each; and let us further suppose that the quantity sold was disposed of in this way:—13 lbs. to one customer, 3 qrs. to a second, and 7 cwt. to a third. After the sale of the 13 lbs. the number of 1-lb. pieces in the stall was (18 – 13=) 5. After the sale of the 3 qrs., which must have been cut off one of

cwt.	qrs.	lbs.
10	2	18
7	3	13
2	3	5

* Simple Subtraction must here be understood as including Simple Addition.

the 1-cwt. pieces, the butcher had (1 cwt. − 3 qrs. =) 1 qr. more, and 1 cwt. less, than at first; so that the number of quarters then was (2 + 1 =) 3. After the sale of the 7 cwt. the butcher had (7 + 1 =) 8 cwt. less than at first—that is, had (10 − 8 =) 2 cwt. remaining. The required remainder is thus found to be 2 cwt. 3 qrs. 5 lbs.

80. Rule for Compound Subtraction: Write the subtrahend under the minuend, in such a way that every two numbers of the same denomination shall stand in the same vertical column. Draw a horizontal line to separate the subtrahend from the remainder. Then, take the lowest denomination in the subtrahend from that in the minuend, and set down the result as the lowest denomination in the required remainder. Should this subtraction not be possible, find the difference between the lowest denomination in the subtrahend and a unit of the next higher denomination; to this difference add the lowest denomination in the minuend; write the result as the lowest denomination in the required remainder; and carry 1 to the next denomination in the subtrahend. Treat each of the other denominations in the same way—always taking the "carried" 1 into account, when there is 1 to carry.

COMPOUND MULTIPLICATION.

81. Multiplication is called COMPOUND when the multiplicand is a compound number.

82. Compound Multiplication is performed by means of Simple Multiplication and Ascending Reduction.

EXAMPLE I.—Multiply £59 17*s.* 8½*d.* by 11.

Multiplying the ½*d.* by 11, we obtain 11 halfpence: this amount, when divided by 2 (the number of halfpence in a

penny), becomes $5\frac{1}{2}d.$, of which we set down the $\frac{1}{2}d.$, and "carry" the $5d.$ Multiplying the $8d.$ by 11, we obtain $88d.$, to which we add the carried $5d.$: we then have 93 pence, and this amount, when divided by 12 (the number of pence in a shilling), becomes $7s.$ $9d.$, of which we set down the $9d.$, and carry the $7s.$ Multiplying the $17s.$ by 11, and adding the carried $7s.$ to the product, we obtain $194s.$: this amount, when divided by 20 (the number of shillings in a pound), becomes £9 $14s.$, of which we set down the $14s.$, and carry the £9. Lastly, multiplying the £59 by 11, and adding the carried £9 to the product, we obtain £658 So that the required product is £658 $14s.$ $9\frac{1}{2}d.$

£	s.	d.
59	17	$8\frac{1}{2}$
		11
658	14	$9\frac{1}{2}$

Note.—In the absence of Reduction, this result (obtained by means of Simple Multiplication) would appear under the form £649 $177s.$ $88\frac{11}{2}d.$ And £658 $14s.$ $9\frac{1}{2}d.$ would be obtained—although in a round-about way—by means of Compound Addition if £59 17s. $8\frac{1}{2}d.$ were taken as addend 11 times.

Example II.—There are 9 silver spoons, each weighing 1 oz. 6 dwts. 11 grs.: how much do they all weigh?

Here we have to multiply 1 oz. 6 dwts. 11 grs. by 9. Multiplying the 11 grs. by 9, we obtain 99 grs.: this number, when divided by 24 (the number of grains in a pennyweight), becomes 4 dwts. 3 grs., of which we set down the 3 grs., and carry the 4 dwts. Multiplying the 6 dwts. by 9, and adding the carried 4 dwts. to the product, we obtain 58 dwts.: this number, when divided by 20 (the number of pennyweights in an ounce), becomes 2 oz. 18 dwts., of which we set down the 18 dwts., and carry the 2 oz. Multiplying the 1 oz. by 9, and adding the carried 2 oz. to the product, we obtain 11 oz. So that the required product is 11 oz. 18 dwts. 3 grs.

oz.	dwts.	grs.
1	6	11
		9
11	18	3

83. Rule for Compound Multiplication: Multiply the several denominations, successively,—beginning with the lowest,—by the multiplier, and reduce as in Compound Addition; taking care to add to each product the number "carried" from the preceding product.

NOTE 1.—When the multiplier exceeds 12, but is resolvable into a pair of factors, neither of which is greater than 12, we proceed as in Simple Multiplication —first, multiplying the multiplicand by one of the factors, and then multiplying the resulting product by the other factor. Thus, if we wanted to find the rent of a farm containing 72 acres, at £3 17*s.* 5½*d.* per acre, we could imagine the farm divided into 9 fields of 8 acres each, in which case the rent of one field would be £3 17*s.* 5½*d.*×8=£30 19*s.* 8*d.*, and the rent of the whole 9 fields £30 19*s.* 8*d.*×9=£278 17*s.* 0*d.*

£	s.	d.
3	17	5½
		8
30	19	8
		9
278	17	0

NOTE 2.—The multiplier, when greater than 12, and not resolvable into factors, is sometimes broken up (as in Simple Multiplication) into the parts which its digits individully represent—the rent of 365 acres of land, for instance, at £3 17*s.* 5½*d.* per acre, being calculated as follows:—

£	s.	d.			
3	17	5½	=	rent of	1 acre.
		10			
38	14	7	=	,, ,,	10 acres.
		10			
387	5	10	=	,, ,,	100 ,,
		3			
1161	17	6	=	,, ,,	300 ,,
232	7	6*	=	,, ,,	60 ,,
19	7	3½	=	,, ,,	5 ,,
1413	12	3½	=	,, ,,	365 ,,

In many cases, however, the pupil will find it less troublesome to proceed according to the Rule. Thus, to return to the last exercise:—

365×½*d.*=365 *halfpence* =	182½*d.*		
365×5*d.*=	1825*d.*		
	2007½*d.*=	167*s.* 3½*d.*	
365×17*s.*=		6205*s.* 0*d.*	£ s. d.
		6372*s.* 3½*d.*=	318 12 3½
365×£3.=			1095 0 0
365×£3 17*s.* 5½*d.*=			1413 12 3½

* The rent of 60 acres is obtained from the multiplication of £38 14*s.* 7*d.* (the rent of 10 acres) by 6.

COMPOUND DIVISION.

84. Division is called COMPOUND when the dividend, or the divisor, or the resulting quotient is a compound number; or when—all three being simple numbers—the dividend and the divisor (whilst of the *same kind*) are of different denominations.

We perform an operation in Compound Division when we divide £23 by 8, and write the quotient under the form £2 17*s.* 6*d.*; also, when we find how many sums of 15*s.* each are contained in £3.

85. Compound Division is performed by means of Simple Division and Descending Reduction.

EXAMPLE I.—If £658 14*s.* 9½*d.* were divided equally amongst 11 persons, how much would each receive?

Here we have to divide £658 14*s.* 9½*d.* by 11. The division of £658 by 11 gives £59 for quotient, and £9 for remainder. This remainder, when reduced, becomes (9×20=) 180 shillings, to which we add the 14 shillings in the dividend. Altogether, we then have (180+14=) 194 shillings, the division of which by 11 gives 17*s* for quotient, and 7*s.* for remainder. This second remainder, when reduced, becomes (7×12 =) 84 pence, to which we add the 9 pence in the dividend. We then have, altogether, (84+9=) 93 pence, the division of which by 11 gives 8*d.* for quotient, and 5*d.* for remainder. This last remainder, when reduced, becomes (5×4=) 20 farthings, to which we add the 2 farthings (½*d.*) in the dividend. Altogether, there then are (20+2=) 22 farthings, the division of which by 11 gives 2 farthings or ½*d.* for quotient, and no remainder. We thus find that each person's share would be £59 17*s.* 8½*d.*

	£	s.	d.
11)	658	14	9½
	59	17	8½

This result could be obtained, although in a very roundabout way, by Compound Subtraction—in other words, by Simple Subtraction and Descending Reduction. Thus, taking £11 as often as possible from £658, we should find that £1 could be given 59 different times to each of the 11 persons, and that the number of pounds then remaining would be 9. Taking 11*s.* as often as possible from 194*s.* (£9 14*s.*), we should next find that 1*s.* could be given 17 different times to each person, and that the number of shillings then remaining would be 7. Taking 11*d.* as often as possible from 93*d.* (7*s.* 9*d.*), we should next find that 1*d.* could be given 8 different times to each

person, and that the number of pence then remaining would be 5. Lastly, taking 11 farthings as often as possible from 22 farthings ($5\frac{1}{2}d.$), we should find that $\frac{1}{4}d.$ could be given 2 different times to each person, and that nothing would then remain to be distributed: each person's share being thus found to be, as before, £59 17*s.* $8\frac{1}{2}d.$

EXAMPLE II.—How many lengths of 2 ft. 8 in. each could be cut off a ribbon 17 yds. long?

In 2 ft. 8 in. there are ($2\times12+8=$) 32 inches, whilst in 17 yds. there are ($17\times3\times12=$) 612 inches. Dividing 612 by 32, therefore, we find that in 17 yds. there are 19 lengths of 2 ft. 8 in. each, and 4 inches over.

$$\begin{array}{rrrl}
 & & \text{yds.} & \\
 & & 17 & \\
\text{ft.} & \text{in.} & \underline{3} & \\
2 & 8 & 51 & \\
\underline{12} & & \underline{12} & \\
32 &) & 612 & (19 \\
 & & \underline{32} & \\
 & & 292 & \\
 & & \underline{288} & \\
 & & 4 &
\end{array}$$

86. Rule for Compound Division, (1.) when the Divisor is an abstract number: Divide the several denominations successively—beginning with the highest—by the divisor, taking care, when a portion of any denomination is left as "remainder," to include this portion, after having reduced it, in the next (lower) denomination. (2.) When the Divisor is a concrete number, reduce it and the dividend to the lowest denomination which occurs in either, and then find the quotient by Simple Division.

NOTE 1.—When the divisor is an abstract number greater than 12, but resolvable into a pair of factors of which neither exceeds 12, we can proceed as in Simple Division — first dividing the dividend by one of the factors, and afterwards dividing the resulting quotient by the other factor. If, for instance, it were required to divide £365 17*s.* $11\frac{3}{4}d.$ by 72, we could resolve 72 into the factors 8 and 9, and proceed as in the margin.

$$\begin{array}{l}
\quad\;\; \text{£} \quad s. \quad d. \\
8)365 \;\; 17 \;\; 11\tfrac{3}{4} \div 72 \\
\hline
9)45 \;\; 14 \;\; 8\tfrac{3}{4}+\tfrac{7}{4}d. \text{ (1st remainder.)} \\
\hline
\quad 5 \;\; 1 \;\; 7\tfrac{1}{2}+\tfrac{5}{4}d. \text{ (2nd ,,)} \\
\text{True quotient} = \text{£}5\ 1s.\ 7\tfrac{1}{2}d. \\
\text{True remainder} = \tfrac{5}{4}d. \times 8 + \tfrac{7}{4}d. \\
\qquad = \tfrac{47}{4}d. = 11\tfrac{3}{4}d.
\end{array}$$

NOTE 2.—When the divisor is an abstract number greater than 12, and not resolvable into factors, we proceed as directed by the Rule. Thus, if it were required to divide £365 17*s.* 11¾*d.* by 47, the quotient (£7 15*s.* 8¼*d.*) would be obtained in the way shown in the margin.

£ *s.* *d.*	
47)365 17 11¾(£7	395(8*d.*
329	376
36	19
20	4
737(15*s.*	79(¼*d.*
47	47
267	32
235	
32	
12	

THE METRIC SYSTEM OF WEIGHTS AND MEASURES.

"The great diversity of Weights and Measures which has existed in all countries has principally arisen from the lesser communities of which they were originally composed having each adopted its own system. In process of time these lesser communities were amalgamated into separate nations, with whose increase of population and trade the inconvenience of a variety of Weights and Measures soon made itself apparent, and the desire of establishing uniformity arose. France was the first country to relieve itself from its barbarous multiplicity of Weights and Measures, by adopting a uniform system. Louis the XVI., at the recommendation of the Constituent Assembly, invited, by a decree, all the nations of Europe, and particularly the King of Great Britain, to confer respecting the adoption of an international system of Weights and Measures. No response being given to this invitation, France committed the consideration of the subject to some of the most learned men of the age, who devised what is called the Metric system: the most simple, convenient, and scientific system of Weights and Measures in existence."*

The Metric system is so called from the circumstance that it is based upon the METRE, which is the standard of length. The metre is the ten-millionth ($\frac{1}{10,000,000}$) part of the distance —measured upon the meridian of Paris—between the equator and the pole, and is longer by about 3⅓ inches than the English yard.

The standard of superficial measure is a square described

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* Report of Parliamentary Committee on Weights and Measures of United Kingdom: 1862.

upon a length of 10 metres. This standard is called an ARE, and is nearly equal in area to 120 square yards.

The standard of solid measure is a cube whose edges are each a metre in length. This standard is called a STERE, and represents a little more than 35 cubic feet.

The standard of capacity is a cubical vessel whose edges are each one-tenth ($\frac{1}{10}$) of a metre in length. This standard is called a LITRE, and represents about a pint and three-quarters.

The standard of weight is the quantity of distilled water which, at a certain temperature,* would fill a cubical vessel whose edges are each one-hundredth ($\frac{1}{100}$) of a metre in length. This standard is called a GRAMME, and represents a little more than 15 grains.†

Of the standards just mentioned, decimal multiples are taken for higher, and decimal submultiples for lower denominations. In connexion with the names of the standards themselves, the Greek words for 10, 100, and 1,000 are employed, as prefixes, to express the higher denominations; whilst the corresponding Latin words are similarly employed to express the lower denominations. Thus,—*deka*, *hekaton*, and *chilia* being the Greek words for 10, 100, and 1,000, respectively,—a length of 10 metres is called a *deka*-metre; a length of 100 metres, a *hecto*-metre; and a length of 1,000 metres, a *kilo*-metre: whilst—*decem*, *centum*, and *mille* being the Latin words for 10, 100 and 1,000, respectively—one-tenth ($\frac{1}{10}$) of a metre is termed a *deci*-metre; one-hundredth ($\frac{1}{100}$) of a metre, a *centi*-metre; and one-thousandth ($\frac{1}{1,000}$) of a metre, a *milli*-metre. So with regard to the other standards.‡

The following are the details of the Metric system:—

LENGTH.	Myriametre§	=	10,000 Metres	
	Kilometre	=	1,000 „	
	Hectometre	=	100 „	
	Dekametre	=	10 „	
	Metre	=	39·3708 inches	
	Decimetre	=	one-tenth ($\frac{1}{10}$)	of a Metre
	Centimetre	=	one-hundredth ($\frac{1}{100}$)	of a Metre
	Millimetre	=	one-thousandth ($\frac{1}{1,000}$)	of a Metre

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* About 39°, Fahrenheit.

† Even the FRANC, the French standard of value, may be said to be based upon the metre, being equal in weight to 5 grammes. The franc is a silver coin,—of which 9 parts out of 10 are pure silver,—and is equivalent to about 10*d*. British.

‡ The only exceptions are—"*millier*" (1,000,000 grammes) and "*quintal*" (100,000 grammes).

§ From *myria*, the Greek word for 10,000.

NOTE.—The Kilometre, in terms of which long distances are usually expressed, represents nearly 5 furlongs: a Kilometre containing 39,371 inches (nearly), whilst in 5 furlongs there are 39,600 inches.

SURFACE.	Hectare	=	100 Ares
	Dekare	=	10 ,,
	Are	=	119·6033 square yards.
	Centiare	=	one-hundredth ($\frac{1}{100}$) of an Are

NOTE.—The Hectare, in terms of which large areas are usually expressed, represents nearly 2½ English acres: a Hectare containing 11,960 square yards, whilst in 2½ English acres there are 12,100 square yards.

SOLIDITY.	Dekastere	=	10 Steres
	Stere	=	35·317 cubic feet
	Decistere	=	one-tenth ($\frac{1}{10}$) of a Stere

CAPACITY.	Kilolitre	=	1,000 Litres
	Hectolitre	=	100 ,,
	Dekalitre	=	10 ,,
	Litre	=	1·76077 pints
	Decilitre	=	one-tenth ($\frac{1}{10}$) of a Litre
	Centilitre	=	one-hundredth ($\frac{1}{100}$) of a Litre

WEIGHT.	Millier	=	1,000,000 Grammes
	Quintal	=	100,000 ,,
	Myriagramme	=	10,000 ,,
	Kilogramme	=	1,000 ,,
	Hectogramme	=	100 ,,
	Dekagramme	=	10 ,,
	Gramme	=	15·4323487 grains
	Decigramme	=	one-tenth ($\frac{1}{10}$) of a Gramme
	Centigramme	=	one-hundredth ($\frac{1}{100}$) of a Gramme
	Milligramme	=	one-thousandth ($\frac{1}{1,000}$) of a Gramme

NOTE.—The Kilogramme, in terms of which the weights of heavy articles are usually expressed, represents about 2⅕ Avoirdupois pounds: a Kilogramme containing 15,432 grains, whilst in 2⅕ pounds (Avoir.) there are 15,400 grains.

It will be seen that, in the Metric system, there are no Compound Rules—the numbers employed to express lengths, surfaces, &c. being all *simple* numbers. Thus,

2 kilometres 4 hectometres 6 dekametres and 8 metres would be written 2468 metres.

3 decigrammes 5 centigrammes and 7 milligrammes	would be written	grammes. ·357
9 hectares 8 ares and 7 centiares	" " "	ares. 908·07
&c.		&c.

It will also be seen that, in this sytem, Reduction is performed by the mere removal of the decimal point. Thus, for the reduction of kilometres to hectometres, dekametres, or metres, the point is simply removed to the right—one place, two places, or three places, as the case may be; whilst, for the reduction of litres to dekalitres, hectolitres, or kilolitres, the point is removed one, two, or three places—as the case may be—to the left:

67·89 kilometres=678·9 hectometres=6789 dekametres=67890 metres; 2345 litres=234·5 dekalitres=23·45 hectolitres=2·345 kilolitres; &c.

Even the numbers which the French employ in expressing sums of money—that is, "money of account"—are simple numbers: a smaller amount than a franc being always written as so many "centimes," or *hundredths* of a franc. Thus, 234 francs and 56 centimes would be written 234·56 francs.*

The following illustrations will enable the student to appreciate the advantages of the Metric system:—

I.—A vintner sold 3 hectolitres 5 dekalitres 7 litres and 9 decilitres of wine on Monday; 2 dekalitres 4 litres 6 decilitres and 8 centilitres on Tuesday; 8 dekalitres 9 litres and 7 decilitres on Wednesday; 2 hectolitres 4 litres and 6 centilitres on Thursday; 7 dekalitres 6 litres 5 decilitres and 4 centilitres on Friday; and 4 hectolitres 6 dekalitres 8 decilitres and 3 centilitres on Saturday: how much did he sell during the week?

By means of Simple Addition we find the answer to be: 1 kilolitre 2 hectolitres 1 dekalitre 3 litres 7 decilitres and 1 centilitre; in other words, 1,213 litres and 71 centilitres.

$$\begin{array}{r} 357{\cdot}9 \\ 24{\cdot}68 \\ 89{\cdot}7 \\ 204{\cdot}06 \\ 76{\cdot}54 \\ 460{\cdot}83 \\ \hline 1213{\cdot}71 \end{array}$$

* There is no special name for a sum of 10, or 100, or 1,000 francs; just as there is no special name for £10, or £100, or £1,000. The French have a coin called a "decime," equal in amount to 10 centimes or the tenth part of a franc, and of about the same value, therefore, as the British penny. The number of decimes is expressed by the figure in the first decimal place.

II.—Off a piece of cloth measuring 5 dekametres 6 decimetres and 7 centimetres, a draper cut a length of 8 metres and 9 centimetres; how much remained?

By means of Simple Subtraction we find the answer to be: 4 dekametres 2 metres 5 decimetres and 8 centimetres; in other words, 42 metres and 58 centimetres.

$$\begin{array}{r} 50{\cdot}67 \\ 8{\cdot}09 \\ \hline 42{\cdot}58 \end{array}$$

III.—A farm is divided into 4 fields, each containing 6 hectares 5 dekares 4 ares and 32 centiares; what is the area of the farm?

By means of Simple Multiplication we find the required area to be: 26 hectares 1 dekare 7 ares and 28 centiares; in other words, 2,617 ares and 28 centiares.

$$\begin{array}{r} 654{\cdot}32 \\ 4 \\ \hline 2617{\cdot}28 \end{array}$$

IV.—A quantity of silver, weighing 5 hectogrammes 9 dekagrammes 2 grammes 5 decigrammes and 9 centigrammes, was made into 6 spoons of equal weight; what did each spoon weigh?

By means of Simple Division we find the answer to be: 9 dekagrammes 8 grammes 7 decigrammes 6 centigrammes and 5 milligrammes; in other words, 98 grammes and 765 milligrammes.

$$\begin{array}{r} 6)592{\cdot}59 \\ \hline 98{\cdot}765 \end{array}$$

V.—The price of a metre of cloth being 18 francs and 60 centimes, what would a length of 7 metres and 4 decimetres cost?

Multiplying 18·6 by 7·4, we find the answer to be 137 francs and 64 centimes.

$$\begin{array}{r} 18{\cdot}6 \\ 7{\cdot}4 \\ \hline 744 \\ 1302 \\ \hline 137{\cdot}64 \end{array}$$

VI.—The rent of a farm containing 13 hectares 5 dekares 7 ares and 90 centiares is 3,394 francs and 75 centimes; what is the rent per are?

Dividing 3,394·75 by 1,357·9, we find the answer to be 2 francs and 50 centimes.

$$\begin{array}{r} 1357{\cdot}9)3394{\cdot}75 \\ \hline 2{\cdot}5 \\ 13579)33947{\cdot}5 \\ 27158\phantom{{\cdot}5} \\ \hline 67895 \\ 67895 \\ \hline \end{array}$$

The Metric system of Weights and Measures has been introduced into most of the countries of Europe; and, except in the case of the Stere and its decimal multiples and submultiples, the

adoption of the system has been rendered "permissive" (but not made compulsory) throughout the British Empire by an Act of Parliament passed in 1864. It is to be feared, however, that, in the absence of a decimal system of British *money*, this Act will not be productive of much practical good.

The issuing of the *florin* or *two-shilling piece* was a step in the direction of a decimal system of money: but there are still required—(1.) a coin, which it has been proposed to call a *cent*, equal in amount to one-tenth of a florin; and (2.) a coin, which it has been proposed to call a *mill*, equal in amount to one-tenth of a cent. The cent would be a silver coin, not quite as large as the present three-penny piece; whilst the mill would be a bronze coin, a little less in size than the present farthing. The coins of account would then stand thus:—

10 Mills	=	1 Cent
10 Cents	=	1 Florin
10 Florins	=	1 Pound

So that—a pound being regarded as the standard—3 pounds 5 florins 7 cents and 9 mills would be represented by the simple number £3·579.

Conversion of Shillings and Pence into a Decimal of a Pound, and vice versa.

A tenth of a pound being a florin (2 shillings), any number of tenths will represent the same number of florins:—

£·1=2*s.*; £·2=4*s.*; £·3=6*s.*; £·4=8*s.*; £·5=10*s.*; &c.

As, when the unit is a pound, 5 in the first decimal place represents 5 florins or 10 shillings, 5 in the second decimal place—that is, 5 hundredths—represents 10 times a smaller amount, or 1 shilling: £·5=10*s.*; £·05=1*s.*

A pound, which is worth 960 farthings, being divisible into 1,000 thousandths (or "mills"), we have 960 farthings=1,000 thousandths, and (dividing by 40) 24 farthings=25 thousandths. So that, in writing farthings as thousandths, we add 1 for every 24; whilst, in writing thousandths as farthings, we reject 1 for every 25.

farthings.		thousandths.
40)960	=	1,000
24	=	25

Example I.—Express 14*s.* 5½*d.* as a decimal of a pound.

In the given amount there are 7 florins, for which we write seven tenths of a pound (£·7). In the remainder of the amount (5½*d.*) there are 22 farthings, for which—the number being

nearly 24—we write 23 (22+1) thousandths. The required decimal is thus found to be £·723.

EXAMPLE II.—Express 9*s*. 10¾*d*. as a decimal of a pound.

In 9 shillings there are 4 florins and 1 shilling; for the 4 florins we write 4 tenths of a pound, and for the 1 shilling 5 hundredths of a pound: 9*s*.=£·45. In the remainder of the amount (10¾*d*.) there are 43 farthings, for which—the number being nearly 48—we write 45 (43+2) thousandths. The required decimal is thus found to be (·45+·045=) £·495.

87. **Rule for the Conversion of Shillings and Pence into a Decimal of a Pound:** Set down as many tenths as there are florins (or two-shilling pieces); and if there be a shilling remaining, write 5 in the second decimal place. Reduce the remainder of the amount to farthings, for which write the same number of thousandths—adding 1 for every 24.

EXAMPLE III.—Express £·723 in shillings and pence.

For the 7 tenths (£·7) we take 7 florins or 14 shillings. There then remain 23 thousandths, for which—the number being nearly 25—we take 22 (23−1) farthings, or 5½*d*. The required amount is thus found to be 14*s*. 5½*d*.

EXAMPLE IV.—Express £·495 in shillings and pence.

For the 4 tenths (£·4) we take 4 florins or 8 shillings, and for 5 of the 9 hundredths an additional shilling. There then remain (9−5=) 4 hundredths and 5 thousandths, or 45 thousandths, for which—the number being nearly 50—we take 43 (45−2) farthings, or 10¾*d*. The required amount is thus found to be (8*s*.+1*s*.+10¾*d*.=) 9*s*. 10¾*d*.

88. **Rule for the Conversion of a Decimal of a Pound into Shillings and Pence:** Set down twice as many shillings as there are tenths, and an additional shilling if the number of hundredths be not less than 5. Express the remainder of the decimal in thousandths, for which write the same number of farthings—rejecting 1 for every 25.

MEASURES AND MULTIPLES:

Prime and Composite Numbers: &c.

89. Two numbers are said to be related as MEASURE and MULTIPLE when the smaller is contained an exact number of times in the larger—in other words, when the division of the larger by the smaller leaves no remainder. Of two numbers so related, the smaller is called the "measure," and the larger the "multiple."

Thus, we say that 7 is a *measure* of 21, or that 21 is a *multiple* of 7: because 7 is contained an exact number of times in 21. The numbers 8 and 20, however, are *not* related as measure and multiple—the division of 20 by 8 leaving a remainder.

Every number* has at least two measures—itself and unity.

90. A number whose *only* measures are itself and unity is called a PRIME number.

The following are "prime" numbers: 1, 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, &c.

91. A number which is *not* "prime"—in other words, a number which has one or more measures besides itself and unity—is called a COMPOSITE number.

The following are "composite" numbers: 4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21, &c.

92. Every number is either ODD or EVEN. The "even" numbers are those of which 2 is a measure—such as 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, &c. The "odd" numbers are those of which 2 is not a measure—such as 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, &c. Of the odd numbers, some—such as 1, 3, 5, 7, 11, 13—are prime; whilst others—such as 9, 15, 21, 25, 27, 33—are composite. Of the even numbers, all are composite except 2, which, having no measure except itself and unity, is a prime number.

---

* That is, every number greater than unity.

When considered with reference to two or of its multiples, a number is called a COMMON ıre of those multiples.

ı, we say that 7 is a "common" measure of 21 and 28; ı is a "common" measure of 20, 30, and 40; &c.

By the GREATEST common measure of two or ıumbers is meant—the largest (or "greatest") re which those numbers have in common.

ng 24 and 36, for example, we see that the measures of 1, 2, 3, 4, 6, 8, 12, and 24; ı measures of 36 are 1, 2, 3, 12, 18, and 36; that the numhich measure 24 but do *not* ı 36 are 8 and 24; that the 's which measure 36 but do ısure 24 are 9, 18, and 36; nsequently, (8, 24, 9, 18, and g rejected,) the *common* mea- 24 and 36 are 1, 2, 3, 4, 12; and that the GREATEST ı measure is 12.

Measures of 24.	*Common* measures of 24 and 36.	Measures of 36.
24		36
1	1	1
2	2	2
3	3	3
4	4	4
6	6	6
~~8~~		~~9~~
12	12	12
~~24~~		~~18~~
		~~36~~

GREATEST common measure = 12.

Two numbers which have no common measure ; is, no common measure greater than unity— id to be prime *to one another*, or *relatively* Such numbers, however, are not necessarily *in themselves*, or *absolutely* prime.

, 15 and 22—having no common measure (except —are prime to one another, or "relatively" prime; but 15 nor 22 is a prime number, 15 being measured by 3 whilst 22 is measured by 2 and 11.

To find the Greatest Common Measure of two ərs: Divide the larger number by the smaller, ' there be a remainder, divide the smaller ər by it; if there be a second remainder, the first remainder by it; if there be a third ıder, divide the second remainder by it; and

so on. Continue the process until nothing remai and the last divisor—that which leaves no r mainder—will be the greatest common measure.*

EXAMPLE I.—Find the greatest common measure of 92 a 438.

Dividing 438 by 92, we obtain 70 for remainder. Dividing 92 by 70, we obtain 22 for second remainder. Dividing 70 by 22, we obtain 4 for third remainder. Dividing 22 by 4, we obtain 2 for fourth remainder. Lastly, dividing 4 by 2, we find that there is no remainder. So that, according to the Rule, 2 is the greatest common measure required.

```
92)438(4
   368
   ———
    70)92(1
       70
       ——
       22)70(3
          66
          ——
           4)22(5
             20
             ——
              2)4
                4
                —
                0
```

EXAMPLE II.—Find the greatest common measure of and 175.

Proceeding as in the margin, we find that the "last divisor" is 1, which, therefore, according to the Rule, is the greatest common measure required. So that 46 and 175 are prime to one another.

```
46)175(3
   138
   ———
    37)46(1
       37
       ——
        9)37(4
          36
          ——
           1)9(9
             9
             —
             0
```

---

* The following rendering of the Rule occurs in some of the treatises on Arithmetic, and deserves to be recorded:—

"The greater by the less divide;
"The less, by what remains beside;
"The last divisor, still, again,
"By what remains—till nought remain:
"And what divides and leaveth nought
"Will be the common measure sought."

AMPLE III.—Find the greatest common measure of 37 85.

$$37)185(5$$
$$185$$
$$0$$

the division of 185 by 37 leaves no remainder, 37 is the st common measure required. For, measuring both itself 85, 37 is a common measure; and it is obviously im-le for two numbers, one of them being 37, to have a r common measure than 37.

son of the Rule.—In examining the reason of the fore- Rule, we must bear the following two facts in mind:

A number which measures another will measure any *le* of that other.

A number which measures two others will measure both *m* and the *difference* of those others.

se facts are rendered intelligible by a little reflection:—

Taking 21, for instance, which contains an exact er of *sevens*, we see that the double, or the treble, or the uple, &c. of 21 must also contain an exact number of . Again: if a debt, consisting of a certain number of gs, could be paid in instalments of (say) 5 shillings each, bvious that a debt twice, or 3 times, or 4 times, &c. as could also be paid in 5-shilling instalments.

eral Demonstration.—Let m measure a: to prove that m res ax, a multiple of a. As sures a, the division of a by give (say) the quotient q, and no remainder. We therefore $a=mq$; (multiplying by x) qx; and (dividing by m) $=(mqx\div m=)\ qx$. It thus rs that m measures ax—the n of ax by m leaving no nder.

$$m)\ a\ (q$$
$$mq$$
$$0$$
$$a=mq$$
$$ax=mqx$$
$$ax\div m=(mqx\div m=)qx$$

As 33 and 12 contain, each, an exact number of *threes*, both 12 and 33—12 must also contain, each, an exact number *es*. Thus, there being 11 *threes* in 33, and 4 *threes* in 12, must be (11+4=) 15 *threes* in 33+12, and (11—4=) *es* in 33—12. Again: if there were two papers of pins, per containing a larger number than the other, but each ning an exact number of rows of (say) 12 pins each, it is t that the two papers, taken together, would contain an number of such rows. It is also evident that, if a row orn off the smaller paper, and a row off the larger; then, r row off the smaller, and another off the larger; a third row off the smaller, and a third off the larger;

and so on, until the smaller paper would have been exhausted—there would remain, of the larger paper, the difference between what it contained and what the smaller paper contained; and this difference would be in the form of *one or more rows* of 12 each.

General demonstration.—Let x and y be two numbers, of which x is the greater, and let both be measured by m: to prove that $x+y$ and $x-y$ are also measured by m.

$$\begin{array}{l|ll} m) & x & y \\ \hline & x' & y' \end{array}$$

Let x and y be divided by m, and let the resulting quotients (there can be no remainders) be x' and y', respectively. We then have $x=mx'$, and $y=my'$; $x+y=mx'+my'=m(x'+y')$, and $x-y=mx'-my'=m(x'-y')$; $(x+y)\div m=m(x'+y')\div m=x'+y'$, and $(x-y)\div m=m(x'-y')\div m=x'-y'$. It thus appears that m is contained an exact number of times in $x+y$, and also in $x-y$: in other words, that both $x+y$ and $x-y$ are measured by m.

$$x=mx';\ y=my';$$
$$x+y=mx'+my'=m(x'+y');$$
$$x-y=mx'-my'=m(x'-y');$$
$$(x+y)\div m=x'+y';$$
$$(x-y)\div m=x'-y'.$$

Let us now return to Example I.:—Beginning at the bottom, we see that the "last divisor," 2, being a measure of 4, measures 20, a multiple of 4; that, measuring 20 and itself, 2 measures 22—the sum of 20 and itself; that, measuring 22, 2 measures 66—a multiple of 22; that, measuring 66 and 4 (the last dividend), 2 measures 70—the sum of 66 and 4; that, measuring 70 and (as we have already seen) 22, 2 measures 92—the sum of 70 and 22; that, measuring 92, 2 measures 368—a multiple of 92; and that, measuring 368 and (as we have already seen) 70, 2 measures 438—the sum of 368 and 70.

```
92)438(4
   368
   ---
    70)92(1
       70
       --
       22)70(3
          66
          --
           4)22(5
             20
             --
              2)4(2
                4
                -
                0
```

So that 2 is, at all events, a common measure of 92 and 438.

It is also the greatest common measure. For, if 92 and 438 were both measured by a larger number than 2, this larger number would measure $(92\times 4=)$ 368, and $(438-368=)$ 70, and $(92-70=)$ 22, and $(22\times 3=)$ 66, and $(70-66=)$ 4, and $(4\times 5=)$ 20, and $(22-20=)$ 2. But a number larger than 2 cannot measure 2: neither, therefore, can a number larger than 2 be a common measure of 92 and 438. So that 2 is not only

a common measure, but the greatest common measure, of 92 and 438.

General demonstration.—Let x and y be two numbers, of which y is the greater. Let the division of y by x give a for quotient and $(y-ax=)$ b for remainder; the division of x by b, c for quotient and $(x-bc=)$ d for remainder; the division of b by d, e for quotient and $(b-de=)$ f for remainder; and the division of d by f, g for quotient, and no remainder: to prove that f is the greatest common measure of x and y.

$$\begin{array}{llllll}
x)\,y\,(a & & & & & \\
\underline{\;ax\;} & & & & & \\
\;\;b)\,x\,(c & & & & & \\
\;\;\;\;\underline{\;bc\;} & & & & & \\
\;\;\;\;\;\;d)\,b\,(e & & & & & \\
\;\;\;\;\;\;\;\;\underline{\;de\;} & & & & & \\
\;\;\;\;\;\;\;\;\;\;f)\,d\,(g & & & & & \\
\;\;\;\;\;\;\;\;\;\;\;\;\underline{\;fg\;} & & & & & \\
\;\;\;\;\;\;\;\;\;\;\;\;\;\;0 & & & & &
\end{array}$$

The division of d by f leaving no remainder, f is a measure of d, and therefore of de, and (f measuring itself) of $(de+f=)$ b, and of bc, and of $(bc+d=)$ x, and of ax, and of $(ax+b=)$ y. So that f is a common measure of x and y.

It is also the greatest common measure. For, if x and y were both measured by a larger number than f, this larger number would measure ax, and $(y-ax=)$ b, and bc, and $(x-bc=)$ d, and de, and $(b-de=)$ f. But a larger number than f cannot measure f: neither, therefore, can a larger number than f be a common measure of x and y. Consequently f, which has been shown to be a common measure, is the greatest common measure of x and y.

NOTE.—*The greatest common measure of two numbers is measured by every other common measure.* For we have just seen that any number which measures x and y must measure their greatest common measure, f; and in the case of Ex. I. we saw that no number could measure 92 and 438 without measuring their greatest common measure, 2. The common measures of 24 and 36, as we have already seen, are 1, 2, 3, 4, 6, and 12; and it will be observed that 12, the greatest common measure, is measured by each of the others.

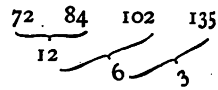
97. To find the Greatest Common Measure of *more than two* numbers: Find the greatest common measure—first, of two of the numbers; next, of this common measure and a third number; then, of this new common measure and a fourth number; and so on, until all the numbers have been taken into account. The last common measure will be the greatest common measure required.

EXAMPLE.—Find the greatest common measure of 72, 84, 102, and 135.

The greatest common measure of 72 and 84 is 12; of 12 and 102, 6; and of 6 and 135, 3. So that, according to the Rule, 3 is the greatest common measure required:—

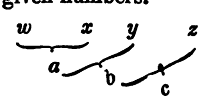
72)84(1	12)102(8	6)135(22
72	96	132
—	—	—
12)72(6	6)12(2	3)6(2
72	12	6
—	—	—
0	0	0

Reason of the Rule.—Measuring 135 and 6, 3 measures 135, 102, and 12—the last two numbers being multiples of 6; and, measuring 12, 3 measures 72 and 84, which are multiples of 12. So that 3 is a common measure of 72, 84, 102, and 135.



It is also the greatest common measure, as we shall find if we remember that (Note, p. 109) "the greatest common measure of two numbers is measured by every other common measure." For, if a larger number than 3 were a common measure of 72, 84, 102, and 135, this larger number—measuring 72 and 84—would measure 12, the greatest common measure of 72 and 84; and—measuring 12 and 102—would measure 6, the greatest common measure of 12 and 102; and, lastly,—measuring 6 and 135,—would measure 3, the greatest common measure of 6 and 135. But a larger number than 3 cannot measure 3: neither, therefore, can a larger number than 3 be a common measure of 72, 84, 102, and 135. So that 3 is not only a common measure, but the greatest common measure, of the given numbers.

General demonstration.—Let w, x, y, and z be four numbers, and let the greatest common measure of w and x be a; of a and y, b; and of b and z, c: to prove that c is the greatest common measure of w, x, y, and z.



Measuring z and b, c measures z, y, and a—the last two numbers being multiples of b; and, measuring z, y, and a, c measures z, y, x, and w—the last two numbers being multiples of a. So that c is a common measure of the four given numbers.

It is also the greatest common measure. For, if a larger number than c were a common measure, this larger number—measuring w and x—would measure a, the greatest common

of *w* and *x*; and—measuring *a* and *y*—would measure
reatest common measure of *a* and *y*; and, lastly,—
ng *b* and *z*,—would measure *c*, the greatest common
of *b* and *z*. But a larger number than *c* cannot
c: neither, therefore, can a larger number than *c* be a
measure of *w*, *x*, *y*, and *z*. Consequently *c*, which
n proved to be a common measure, is the greatest
measure of the given numbers.

When considered with reference to two or
f its measures, a number is called a COMMON
le of those measures.

we say that 24 is a "common" multiple of 4 and 6;
is a "common" multiple of 4, 6, and 9; &c.

By the LEAST common multiple of two or
numbers is meant—the smallest (or "least")
le which those numbers have in common.

g 12 and 8, for example,
that the multiples of 12
itself, 24, 36, 48, 60, 72,
108, 120, 132, 144, &c.;
e multiples of 8 are: 8
5, 24, 32, 40, 48, 56, 64,
88, 96, 104, 112, 120,
5, 144, &c.; that the num-
ich, although multiples of
not multiples of 8, are: 12,
84, 108, 132, &c.; that
bers which, although mul-
8, are *not* multiples of 12,
16, 32, 40, 56, 64, 80, 88,
2, 128, 136, &c.; that the
multiples of 12 and 8 are:
72, 96, 120, 144, &c.; and
LEAST common multiple

Multiples of 12.	Common multiples of 12 and 8.	Multiples of 8.
12		8
—		—
~~12~~	24	~~8~~
24	48	~~16~~
~~36~~	72	24
48	96	~~32~~
~~60~~	120	~~40~~
72	144	48
~~84~~	&c.	~~56~~
96		~~64~~
~~108~~		72
120		~~80~~
~~132~~		~~88~~
144		96
&c.		~~104~~
		~~112~~
		120
		~~128~~
		~~136~~
		144
		&c.

LEAST common multiple=24.

To find the Least Common Multiple of two
rs: Divide the product of the numbers by
greatest common measure.*

PLE.—What is the least common multiple of 12 and 20?

---

* is most easily effected when we divide one of the numbers by
test common measure, and multiply the other number by the
quotient.

The greatest common measure of the two numbers being 4, the least common multiple is $\frac{12 \times 20}{4} = 60$.

NOTE.—The least common multiple of two numbers which are prime to one another is their product, the greatest common measure being unity. Thus, the least common multiple of 15 and 22—the greatest common measure being 1—is $\left(\frac{15 \times 22}{1} = \right) 15 \times 22$.

Reason of the Rule.—In examining the reason of the foregoing Rule, we must remember that—

Every other common multiple of two numbers is a multiple of the least common multiple.

Thus, the least common multiple of 12 and 8 is 24, and every other common multiple (48, 72, 96, 120, 144, &c.) is a multiple of 24. In like manner, the least common multiple of 15 and 10 is 30, and every other common multiple (60, 90, 120, 150, 180, &c.) is a multiple of 30.

$$\left.\begin{matrix} x \\ y \end{matrix}\right\} l) m (q \\ \quad \underline{lq} \\ \quad r$$

Passing from numbers to general symbols, let us put x and y to represent two numbers, l their least common multiple, and m any other common multiple. Now, if m be not a multiple of l, the division of m by l will leave a remainder. Let l be contained q times in m, and let the subtraction of lq from m leave the remainder r, if possible. Then, x and y, being measures of l, are measures of lq—a multiple of l; x and y are also measures of m; consequently x and y are measures of r—the difference between lq and m. In other words, r is a common multiple of x and y. But this is impossible, the *least* common multiple of x and y being l, and r (the "remainder") being less than l (the "divisor"). So that the division of m by l can leave no remainder: in other words, m is a multiple of l.

Returning to the last example, and remembering that the product of two numbers is a common multiple of those numbers, we see that 12×20 is a common multiple of 12 and 20. Now, 12×20, if not the least common multiple, must be a multiple of the least, which, consequently, will be obtained from the division of 12×20 by some whole number. If we try 10 for divisor, we shall find that $\frac{12 \times 20}{10}$ is a multiple of 12, but not of 20; 10 being a measure of 20, but not of 12. For, 10 being a measure of 20, $\frac{20}{10}$ is equivalent to a whole number (2); so that $\frac{12 \times 20}{10}$, which is the product of 12 by $\frac{20}{10}$, is a multiple of 12. But as 10 is not a measure of 12, $\frac{12}{10}$ is not equivalent to a whole number; so that $\frac{12 \times 20}{10}$, which is the product of 20

by $\frac{12}{10}$, is not a multiple of 20. If we next try 6 for divisor, we shall find that $\frac{12 \times 20}{6}$ is a multiple of 20, but not of 12; 6 being a measure of 12, but not of 20. For, 6 being a measure of 12, $\frac{12}{6}$ is equivalent to a whole number (2); so that $\frac{12 \times 20}{6}$, which is the product of 20 by $\frac{12}{6}$, is a multiple of 20. But as 6 is not a measure of 20, $\frac{20}{6}$ is not equivalent to a whole number; so that $\frac{12 \times 20}{6}$, which is the product of 12 by $\frac{20}{6}$, is not a multiple of 12.

Let us now try 2 for divisor, and we shall find that $\frac{12 \times 20}{2}$ is a common multiple of 12 and 20; 2 being a common measure. For, as 2 measures 20, $\frac{20}{2}$ is equivalent to a whole number (10); so that $\frac{12 \times 20}{2}$, which is the product of 12 by $\frac{20}{2}$, is a multiple of 12. Again: as 2 measures 12, $\frac{12}{2}$ is equivalent to a whole number (6); so that $\frac{12 \times 20}{2}$, which is the product of 20 by $\frac{12}{2}$, is a multiple of 20. It thus appears that the divisor must be a common measure; and upon the principle that—the dividend remaining the same—the quotient is smallest when the divisor is largest, we obtain the least common multiple of 12 and 20 when we divide 12×20 by 4, the greatest common measure.

General demonstration.—Let x and y be any two numbers, and m their greatest common measure: to prove that $\frac{xy}{m}$ is the least common multiple of x and y. As m is a measure of y, $\frac{y}{m}$ is equivalent to a whole number; and $x \times \frac{y}{m}$, or $\frac{xy}{m}$, is a multiple of x. Again: as m is a measure of x, $\frac{x}{m}$ is equivalent to a whole number; and $y \times \frac{x}{m}$, or $\frac{xy}{m}$, is a multiple of y. So that $\frac{xy}{m}$ is a common multiple of x and y.

It is also the least common multiple. For, if not, it must (p. 112) be a multiple of the least. Let $\frac{xy}{m}$ be a times the least common multiple, *if possible*: in other words, let the least common

multiple be $\left(\frac{xy}{m} \div a =\right) \frac{xy}{am}$. Then, because $\frac{xy}{am}$, which is the product of x by $\frac{y}{am}$, is a multiple of x, $\frac{y}{am}$ must be equivalent to a whole number: so that am must measure y. And because $\frac{xy}{am}$, which is the product of y by $\frac{x}{am}$, is a multiple of y, $\frac{x}{am}$ must be equivalent to a whole number: so that am must measure x. It thus appears that am is a common measure of x and y. But this is impossible, the greatest common measure of x and y being m. It follows, therefore, that $\frac{xy}{m}$ is the least common multiple of x and y.

101. To find the Least Common Multiple of *more than two* numbers: Find, first, the least common multiple of two of the numbers; next, the least common multiple of this common multiple and a third number; then, the least common multiple of this new common multiple and a fourth number; and so on, until all the numbers have been taken into account. The last common multiple will be the least common multiple required.

EXAMPLE.—Find the least common multiple of 8, 12, 18, and 27.

The least common multiple of 8 and 12 is 24; of 24 and 18, 72; and of 72 and 27, 216. So that, according to the Rule, 216 is the least common multiple required.

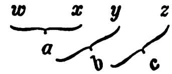
$$\begin{array}{cccc} 8 & 12 & 18 & 27 \\ \multicolumn{2}{c}{\underbrace{\qquad\quad}_{24}} & & \\ & 72 & & \\ & & 216 & \end{array}$$

Reason of the Rule.—As 24 is a common multiple of 8 and 12, 8 and 12 are measures of 24, and therefore of 72—a multiple of 24; 18 also is a measure of 72; consequently 8, 12, and 18, being measures of 72, are measures of 216—a multiple of 72. In other words, 216 is a common multiple of 8, 12, and 18: therefore, being a multiple of 27 also, 216 is a common multiple of the four given numbers.

It is also the least common multiple. For, if a smaller number than 216 were a common multiple of 8, 12, 18, and 27, this smaller number, being a multiple of 8 and of 12, would (p. 112) be *a multiple of 24*—the least common multiple of 8 and 12; and, *being a multiple* of 24 and of 18, would be a multiple of 72—

the least common multiple of 24 and 18; and, being a multiple of 72 and of 27, would be a multiple of 216—the least common multiple of 72 and 27. But a smaller number than 216 cannot be a multiple of 216: neither, therefore, can a smaller number than 216 be a common multiple of 8, 12, 18, and 27. So that 216, which has been shown to be a common multiple, is the least common multiple.

General Demonstration.—Let w, x, y, and z be any four numbers, and let the least common multiple of w and x be a; of a and y, b; and of b and z, c: to prove that c is the least common multiple of w, x, y, and z.



Because w and x measure a, they measure b—a multiple of a; y also measures b; consequently w, x, and y, being measures of b, measure c—a multiple of b. In other words, c is a common multiple of w, x, and y: therefore, being a multiple of z also, c is a common multiple of the four given numbers.

It is also the least common multiple. For, if possible, let a smaller number than c be a common multiple of w, x, y, and z. Then, this smaller number, being a common multiple of w and x, is a multiple of a—the least common multiple of w and x; and, being a common multiple of a and y, is a multiple of b—the least common multiple of a and y; and, being a common multiple of b and z, is a multiple of c—the least common multiple of b and z. But a smaller number than c cannot be a multiple of c: neither, therefore, can a smaller number than c be a common multiple of w, x, y, and z. Consequently c, which has been proved to be a common multiple, is the least common multiple.

102. Another Rule for finding the Least Common Multiple of more than two numbers: Arrange the numbers in a horizontal row, and divide as many as possible by a common measure. Set down, in a second horizontal row, the resulting quotients and the undivided numbers. Treat this and every succeeding row in the same way as the first, and continue the process until a row of numbers shall have been obtained, of which every two are prime to each other. Then, multiply the several divisors and the numbers in the last row together, and the product will be the least common *multiple required.*

EXAMPLE.—Find the least common multiple of 50, 60, 80, 90, and 100.

These numbers are all measured by 10. Dividing by 10, we obtain for quotients 5, 6, 8, 9, and 10. Dividing as many as possible (*i.e.*, 6, 8, and 10) of these quotients by a common measure, 2, and bringing down the undivided numbers, we obtain 5, 3, 4, 9, and 5. Of this row of numbers, we can divide not more than two (the first and the last, or the second and the fourth) by a common measure. Dividing the first and the last by 5, and bringing down the undivided numbers, we obtain 1, 3, 4, 9, and 1. Dividing the second and the fourth of this last row of numbers by 3, and bringing down the undivided number 4,—it is obviously unnecessary to bring down the *ones*,—we obtain 1, 4, and 3: of which numbers every two are prime to each other. So that, according to the Rule, the least common multiple required is $10\times2\times5\times3\times4\times3=3{,}600$.

$$\begin{array}{r|rrrrr}
10) & 50 & 60 & 80 & 90 & 100 \\ \hline
2) & 5 & 6 & 8 & 9 & 10 \\ \hline
5) & 5 & 3 & 4 & 9 & 5 \\ \hline
3) & 1 & 3 & 4 & 9 & 1 \\ \hline
 & & 1 & 4 & 3 &
\end{array}$$

Reason of the Rule.—Upon the principle that (when there is no remainder) a dividend is equal to the product of the divisor by the quotient, we have—

$$\begin{array}{lllll}
50 & = & & 10\times5 & \\
 & & & & 10\times2\times5\times3 \\
60 & = & 10\times6= & 10\times2\times3 & \\
80 & = & 10\times8= & 10\times2\times4 & 10\times2\times5\times3\times4 \\
90 & = & 10\times9= & 10\times3\times3 & 10\times2\times5\times3\times4\times3 \\
100 & = & 10\times10= & 10\times2\times5 & 10\times2\times5\times3\times4\times3
\end{array}$$

Now, the least common multiple of the first two numbers (10×5 and $10\times2\times3$)—the greatest common measure being 10—is $(10\times5)\times(10\times2\times3)\div10=10\times2\times5\times3$. The least common multiple of this common multiple and the third number ($10\times2\times4$)—the greatest common measure being 10×2—is $(10\times2\times5\times3)\times(10\times2\times4)\div10\times2=10\times2\times5\times3\times4$. The least common multiple of this common multiple and the fourth number ($10\times3\times3$)—the greatest common measure being 10×3—is $(10\times2\times5\times3\times4)\times(10\times3\times3)\div10\times3=10\times2\times5\times3\times4\times3$. And the least common multiple of this last *common multiple* and the fifth number ($10\times2\times5$)—the *former number being* a multiple of the latter—is $10\times2\times5\times3\times4$

$\times 3$. So that (§ 101) $10 \times 2 \times 5 \times 3 \times 4 \times 3$ is the least common multiple of 50, 60, 80, 90, and 100.

NOTE.—When two numbers in the same row are related as measure and multiple, we save unnecessary trouble by rejecting the measure. Thus, rejecting 50, which measures 100, and dividing the remaining numbers by 10, we have in the second row 6, 8, 9, and 10. Dividing as many as possible (*i.e.*, 6, 8, and 10) of the numbers in the second row by a common measure, 2, and bringing down the undivided number (9), we have in the third row 3, 4, 9, and 5. Rejecting 3, which measures 9, we find that every two of the remaining numbers are prime to one another. And we then obtain the least common multiple by multiplying 10, 2, 4, 9, and 5 together.

$$\begin{array}{r|rrrrr} 10) & \cancel{50} & 60 & 80 & 90 & 100 \\ \hline 2) & & 6 & 8 & 9 & 10 \\ \hline & & \cancel{3} & 4 & 9 & 5 \end{array}$$

General demonstration.—Before proceeding to examine the general demonstration given below, we must understand the following two facts:

I. A number which is prime to two or more others is prime to their product.

II. If the division of two numbers by a common measure give quotients which are prime to each other, that common measure is the *greatest* common measure of the two numbers.

(I.) The first of these facts is rendered intelligible by the following considerations:

(*a*) A (composite) number is said to be resolved into its prime factors when it is resolved into two or more factors, every one of which is a prime number. The number 30, for instance, when written 3×10, is *not* resolved into its prime factors: for, although 3 is, 10 is not, a prime number. But 30 is resolved into its prime factors when written $3 \times 2 \times 5$; 3, 2, and 5 being all prime numbers.

(*b*) If two numbers, x and y, be so related to each other that no one of the prime factors of x is a factor of y, x and y are prime to one another. For, if x and y be not prime to one another, they have some common measure, m, which must be either a prime or a composite number. Now, m cannot be a prime number: for, if it were, one of the prime factors of x would be a factor of y. Neither can m be a composite number: for, if it were, it would be resolvable into two or more prime factors, each of which would measure x and y, multiples of m; so that in this case, also, one of the prime factors of x would be a factor of y. It thus appears that x and y, *having neither a prime nor a composite number for common measure, are prime to one another.*

(*c*) Let x, y, and z be each prime to w: to prove that $x \times y \times z$ is prime to w. Let x' and x'' be the prime factors of x; y', y'', and y''', the prime factors of y; and z', z'', z''', and z'''', the prime factors of z. We then have $x \times y \times z = x' \times x'' \times y' \times y'' \times y''' \times z' \times z'' \times z''' \times z''''$; and it will be seen

$$\overbrace{x' \times x''}^{x} \times \overbrace{y' \times y'' \times y'''}^{y} \times \overbrace{z' \times z'' \times z''' \times z''''}^{z}$$
$$w$$

that, when written under this latter form, the product of x, y, and z is resolved into its prime factors—because [*a*] resolved into factors every one of which is a prime number. Now, as x is prime to w, neither x' nor x'' is a factor of w. Again: as y is prime to w, no one of the prime factors (y', y'', y''') of y is a factor of w. And as z is prime to w, no one of the prime factors (z', z'', z''', z'''') of z is a factor of w. As, therefore, $x \times y \times z$ and w are so related to each other that no one of the prime factors of the former number is a factor of the latter, $x \times y \times z$ is [*b*] prime to w.

Perhaps this demonstration will be better understood if we substitute 14, 18, and 156 for x, y, and z, respectively; and 55 for w. We shall then have—for x' and x'', 2 and 7, respectively; for y' y'', and y''', 2, 3, and 3, respectively; and for z', z'', z''', and z'''', 2, 2, 3, and 13, respectively.

$$\overbrace{2 \times 7}^{14} \times \overbrace{2 \times 3 \times 3}^{18} \times \overbrace{2 \times 2 \times 3 \times 13}^{156}$$
$$\underbrace{5 \times 11}_{55}$$

(II.) The second fact is easily established. Let x and y be divided by a common measure, m, and let the resulting quotients, x' and y', be prime to one another: to prove that m is the *greatest* common measure of x and y.

$$\begin{array}{c|c} m)\underline{x \quad y} & am)\underline{x \quad y} \\ x' \quad y' & x'' \quad y'' \\ x = mx' & x = amx'' \\ y = my' & y = amy'' \end{array}$$

$$mx' = amx''\,; \quad my' = amy''$$
$$x' = ax''\,; \quad y' = ay''$$
$$x' \div a = x''\,; \quad y' \div a = y''$$

Now, m, if not the greatest common measure, must (Note, p. 109) be a measure of the greatest, which, consequently, must be a multiple of m. Let the greatest common measure of x and y be am, if possible, and let the division of x and y by am give the quotients x'' and y'', respectively. Then, mx' and amx''—being each equal to x—are equal to one another; my' and amy'' also—being each equal to y—are equal to one another. Dividing by m, therefore, we have $x' = ax''$, and $y' = ay''$; from which equality it appears that x' and y' are both measured by a. But this is impossible, x' and y' being prime to each other. So that

x and y can have no greater common measure than m, which, consequently, is the greatest common measure.

As a practical illustration, let us take the numbers 24 and 36. When we divide these numbers by 4, one of their common measures, we find that the resulting quotients, 6 and 9, are not prime to one another, for which reason we conclude that 4 is not the greatest common measure of 24 and 36. When, however, we divide by 12, we find that the resulting quotients, 2 and 3, are prime to one another, for which reason we conclude that 12 is the greatest common measure of 24 and 36.

$$\begin{array}{r|rr} 4) & 24 & 36 \\ \hline & 6 & 9 \end{array} \qquad \begin{array}{r|rr} 12) & 24 & 36 \\ \hline & 2 & 3 \end{array}$$

Let us now take the four numbers w, x, y, and z, and suppose that the first three—but not the whole four—have a common measure. Writing the given numbers in a horizontal row, let us divide the first three (w, x, and y) by a common measure, a, and set down, in a second horizontal row, the resulting quotients (w', x', and y'), and also the undivided number, z. Of the numbers in the second row, let us suppose that the last three—but not the whole four—have a common measure. Let us divide the last three numbers (x', y', and z) by a common measure, b, and set down, in a third horizontal row,—along with the undivided number, w',—the resulting quotients, x'', y'', and z'. Of the numbers in the third row, let us suppose that not more than two—the middle two—have a common measure. Let us divide the middle pair of numbers (x'' and y'') by a common measure, c, and write the resulting quotients (x''' and y'''), as well as the undivided numbers, w' and z', in a fourth horizontal row. Of the numbers in the fourth row, let us suppose that every two are prime to one another; and the least common multiple of w, x, y, and z will be found to be $a\times b\times c\times w''\times x''\times y'''\times z'$.

$$\begin{array}{r|cccc} a) & w & x & y & z \\ \hline b) & w' & x' & y' & z \\ \hline c) & w' & x'' & y'' & z' \\ \hline & w' & x''' & y''' & z' \end{array}$$

Expressing the given numbers in terms of the divisors and the numbers in the last horizontal row, we have—

$$\begin{aligned} w &= && && a\times w' \\ x &= a\times x' &&= a\times b\times x'' &&= a\times b\times c\times x''' \\ y &= a\times y' &&= a\times b\times y'' &&= a\times b\times c\times y''' \\ z &= && && b\times z' \end{aligned}$$

Now, if we can show (I.) that the least common multiple of w and x—or of $a\times w'$ and $a\times b\times c\times x'''$—is $a\times b\times c\times w'\times x'''$; (II.) that the least common multiple of $a\times b\times c\times w'\times x'''$ and y—or of $a\times b\times c\times w'\times x'''$ and $a\times b\times c\times y'''$—is $a\times b\times c\times w'\times x'''\times y'''$; and (III.) that the least common multiple of

$a\times b\times c\times w'\times x'''\times y'''$ and z—or of $a\times b\times c\times w'\times x'''\times y'''$ and $b\times z'$—is $a\times b\times c\times w'\times x'''\times y'''\times z'$, it will follow from what has already been established (§ 101) that $a\times b\times c\times w'\times x'''\times y'''\times z'$ is the least common multiple of w, x, y, and z.

(I.) The greatest common measure of w and x—or of $a\times w'$ and $a\times b\times c\times x'''$—is a. For, when we divide w and x—or $a\times w'$ and $a\times b\times c\times x'''$—by a, we find the resulting quotients, w' and $b\times c\times x'''$, to be prime to each other. Thus, w' is prime to b, because otherwise w' and b would have a common measure, which—measuring b—would measure x', y', and z (multiples of b) as well as w'; so that w', x', y', and z would have a common measure, whereas it has been assumed that not more than three of these numbers (which occur in the second horizontal row) have a common measure. Again: w' is prime to c, because otherwise w' and c would have a common measure, which—measuring c—would measure x'' and y'' (multiples of c), as well as w'; so that w', x'', and y'' would have a common measure, whereas it has been assumed that not more than two of these numbers (which occur in the third horizontal row) have a common measure. Lastly: w' is prime to x''', both numbers occurring in the last horizontal row. So that w'—being prime to b, to c, and to x'''—is (p. 117, I.) prime to $b\times c\times x'''$. As, therefore, the division of w and x by a gives quotient which are prime to each other, a is (p. 117, II.) the greatest common measure of w and x. Consequently, the least common multiple of w and x—or of $a\times w'$ and $a\times b\times c\times x'''$—is $(a\times w')\times(a\times q\times c\times x'')\div a=a\times b\times c\times w'\times x'''$.

(II.) The greatest common measure of $a\times b\times c\times w'\times x'''$ and y—or of $a\times b\times c\times w'\times x'''$ and $a\times b\times c\times y'''$—is $a\times b\times c$. For, when we divide the two numbers under consideration by $a\times b\times c$, we find the resulting quotients, $w'\times x'''$ and y''', to be prime to each other. Thus, as w', x''', and y''' all occur in the last horizontal row, y''' is prime to w', and to x''', and therefore to $w'\times x'''$. Consequently, the least common multiple of $a\times b\times c\times w'\times x'''$ and y, or of $a\times b\times c\times w'\times x'''$ and $a\times b\times c\times y'''$,—in other words, the least common multiple of w, x, and y,—is $(a\times b\times c\times w'\times x''')\times(a\times b\times c\times y''')\div a\times b\times c=a\times b\times c\times w'\times x'''\times y'''$.

(III.) The greatest common measure of $a\times b\times c\times w'\times x'''\times y'''$ and z—or of $a\times b\times c\times w'\times x'''\times y'''$ and $b\times z'$—is b. For, dividing the two numbers under consideration by b, we find the resulting quotients, $a\times c\times w'\times x'''\times y'''$ and z', to be prime to each other. Thus, z' is prime to a, because otherwise z' and a would have a common measure, which—measuring a—would measure w, x, and y (multiples of a), as well as z—a multiple of z'; so that w, x, y, and z would have a common measure, whereas it has been assumed that not more than three of these numbers (which occur in the first horizontal row) have a com-

mon measure. Again: z' is prime to c, because otherwise z' and c would have a common measure, which—measuring c—would measure x'' and y'' (multiples of c), as well as z'; so that x'', y'', and z' would have a common measure, whereas it has been assumed that not more than two of these numbers (which occur in the third horizontal row) have a common measure. Lastly: z' is prime to w', to x''', and to y''',—the whole four numbers occurring in the last horizontal row. So that z'—being prime to a, to c, to w', to x''', and to y'''—is prime to $a\times c\times w'\times x'''\times y'''$. As, therefore, the division of $a\times b\times c\times w'\times x'''\times y'''$ and z—or of $a\times b\times c\times w'\times x'''\times y'''$ and $b\times z'$—by b gives quotients which are prime to each other, b is the greatest common measure of $a\times b\times c\times w'\times x'''\times y'''$ and z. Consequently, the least common multiple of $a\times b\times c\times w'\times x'''\times y'''$ and z, or of $a\times b\times c\times w'\times x'''\times y'''$ and $b\times z'$,—in other words, the least common multiple of w, z, y, and z,—is $(a\times b\times c\times w'\times x'''\times y''')\times(b\times z')\div b = a\times b\times c\times w'\times x'''\times y'''\times z'$:

$$\underbrace{\underbrace{\underbrace{a\times w' \quad a\times b\times c\times x''}_{a\times b\times c\times w'\times x'''} \quad a\times b\times c\times y'''}_{a\times b\times c\times w'\times x'''\times y'''} \quad b\times z'}_{a\times b\times c\times w'\times x'''\times y'''\times z'}$$

FRACTIONAL NUMBERS, OR FRACTIONS.

103. If a unit like one of those which we have hitherto been considering—an INTEGRAL unit, let us now say, for the sake of distinction—were divided into any number of *equal* parts, the parts, regarded as such, would be FRACTIONAL UNITS.

104. The *denomination* of a fractional unit is expressed by one of that class of words called "ordinals,"*—the corresponding "cardinal" indicating how many such units there are in an integral unit. Thus, if an integral unit were divided into

* *There is one exception:* instead of "second," we say half.

2, 3, 4, 5, 6, &c.	*equal* parts, each part—a "fractional unit"—would be a	*half*, *third*, *fourth**, *fifth*, *sixth*, &c.	of the integral unit.

So that a school-boy, after eating 1 *fifth* of a cake, would have 4 *fifths* remaining—a cake being divisible into 5 *fifths*; a farmer, after parting with 3 *eighths* of his farm, would have 5 *eighths* remaining—a farm being divisible into 8 *eighths*; and a draper, after selling 5 *twelfths* of a piece of cloth, would have 7 *twelfths* remaining—the number of *twelfths* into which a piece of cloth is divisible being 12.

105. Two or more fractional units of the *same denomination*—that is, two or more "halves," or two or more "thirds," or two or more "fourths," &c. (as the case may be)—constitute a FRACTIONAL NUMBER, or a FRACTION.

"Three *yards*," "three *shillings*," "three *fifths*," "three *eighths*:" in each of these instances, the *number* is "three," whilst the *unit* is—in the first case, a "yard;" in the second, a "shilling;" in the third, a "fifth;" and in the fourth, an "eighth." The first two numbers are "integral;" the last two, "fractional."

In practice, we cannot avoid regarding a fractional *unit* as a fractional "number;" just as we cannot avoid regarding an integral unit as an integral number. For the present, however, it will be well to bear in mind the distinction between a fractional "unit" and a fractional "number." *Three* fifths ($\frac{3}{5}$), *four* sevenths ($\frac{4}{7}$), *five* ninths ($\frac{5}{9}$), &c. are fractional "numbers;" whilst *one* fifth ($\frac{1}{5}$), *one* seventh ($\frac{1}{7}$), *one* ninth ($\frac{1}{9}$), &c. are fractional "units."

106. In writing a fraction,† we employ two (integral) numbers, which are placed one below the other, and separated by a horizontal line. Of these two numbers—which we speak of as the "terms" of the fraction—the upper, called the *numerator*, indicates how many fractional units the fraction is composed of; whilst the lower number, called the *denominator*, indicates how many such units there

* *Fourths* are often called *quarters*.

† *That is, a simple* fraction. (See pp. 138–9.)

are in an integral unit—in other words, indicates the denomination.

Thus: 3 *fifths* $=\frac{3}{5}$; 4 *sevenths* $=\frac{4}{7}$;
5 *eighths* $=\frac{5}{8}$; 7 *ninths* $=\frac{7}{9}$; &c.

A fraction, therefore, which has (say) 10 for numerator, is composed of 10 fractional units of the *same denomination*—10 *halves*, or 10 *thirds*, or 10 *fourths*, &c., as the case may be; but what the denomination is, we are unable to say without knowing the denominator. On the other hand, a fraction which has 10 for denominator is composed of one or more *tenths*; but the number of tenths we cannot tell without knowing the numerator:—

$\frac{10}{2}$=10 *halves*; $\frac{10}{3}$=10 *thirds*; $\frac{10}{4}$=10 *fourths*; $\frac{10}{5}$=10 *fifths*; &c.
$\frac{1}{10}$= 1 *tenth*; $\frac{2}{10}$= 2 *tenths*; $\frac{3}{10}$= 3 *tenths*; $\frac{4}{10}$= 4 *tenths*; &c.

NOTE.—If the question were asked, How can there be such a fraction as, for instance, 10 *halves* of a penny ($\frac{10}{2}d.$), the number of halves into which a penny is divisible being only 2? the answer would be this: We can imagine any number of pence divided into 2 halves *each*, and the division of each of 5 pence into halves would give the fraction referred to; just as we should obtain 11 *fourths* of a 1-lb. loaf by dividing 3 such loaves into 4 fourths each, and setting aside one of the 12 fourths so obtained.

107. We express a concrete number as a fraction of a larger one of the same kind when—having reduced both numbers to the same denomination—we write the smaller for numerator, and the larger for denominator.

Thus, to express 13 grains as a fraction of a pennyweight, we write 13 for numerator, and 24 (the number of grains in a pennyweight) for denominator; to express 2 ft. 5 in. as a fraction of a yard, we write 29 (the number of inches in 2 ft. 5 in.) for numerator, and 36 (the number of inches in a yard) for denominator; to express $3\frac{1}{2}d.$ as a fraction of 2*s*. 6*d*., we write 7 (the number of halfpence in $3\frac{1}{2}d.$) for numerator, and 60 (the number of halfpence in 2*s*. 6*d*.) for denominator; &c.:—

13 grs.$=\frac{13}{24}$ of a pennyweight; 2 ft. 5 in.$=\frac{29}{36}$ of a yard;
$3\frac{1}{2}d.=\frac{7}{60}$ of 2*s*. 6*d*; &c.

This requires *little explanation*. The number of grains in a *pennyweight being 24*, a grain is the *twenty-fourth* part of a

pennyweight; so that 13 grains must be 13 *twenty-fourths* ($\frac{13}{24}$) of a pennyweight. Again: the number of inches in a yard being 36, an inch is the *thirty-sixth* part of a yard; so that 29 inches, or 2 ft. 5 in., must be 29 *thirty-sixths* ($\frac{29}{36}$) of a yard. In like manner, the number of halfpence in 2*s.* 6*d.* being 60, a halfpenny is the *sixtieth* part of 2*s.* 6*d.*; so that 7 halfpence, or 3½*d.*, must be 7 *sixtieths* ($\frac{7}{60}$) of 2*s.* 6*d.*

108. Of fractional UNITS, the largest is a *half* ($\frac{1}{2}$): of others, a *third* ($\frac{1}{3}$) is larger than a *fourth* ($\frac{1}{4}$); a *fourth* is larger than a *fifth* ($\frac{1}{5}$); a *fifth* is larger than a *sixth* ($\frac{1}{6}$); and so on—the size of the unit decreasing as the denominator increases.

This is rendered evident by the fact that "the greater the number of equal parts into which anything is divided, the smaller the size of each part." If, for instance, a legacy were divided equally between two persons, the share of each would be a *half;* if the legacy were divided equally between three persons, the share of each—a *third*—would be smaller; if the legacy were divided equally between four persons, the share of each—a *fourth*—would be still smaller; and so on.

109. The smaller of two fractional UNITS is contained an exact number of times in the larger when —but only when—the two denominators are related as measure and multiple; and when the denominators *are* so related, the quotient resulting from the division of the multiple by the measure will indicate how many times the smaller fractional unit is contained in the larger.

This fact can be established by means of Simple Division. Taking, for example, the fractional units $\frac{1}{4}$ and $\frac{1}{12}$, whose denominators (4 and 12) are related as measure and multiple, we see that the number of *fourths* in an integral unit is 4, and that the number of *twelfths* is 12. We thus have 4 *fourths*=12 *twelfths*, and (dividing by 4) 1 *fourth*=3 *twelfths*. So that the smaller fractional unit ($\frac{1}{12}$) is contained exactly 3 times in the larger ($\frac{1}{4}$)—3 being, it will be observed, the quotient resulting from the division of one denominator (12) by the other (4).

	fourths.		twelfths.
4)	4	=	12
	1	=	3

Again: taking the fractional units $\frac{1}{5}$ and $\frac{1}{20}$, whose denomi-

nators (5 and 20) are related as measure and multiple, we see that the number of *fifths* in an integral unit is 5, and that the number of *twentieths* is 20. We thus have 5 *fifths*=20 *twentieths*, and (dividing by 5) 1 *fifth*=4 *twentieths*. So that the smaller fractional unit ($\frac{1}{20}$) is contained exactly 4 times in the larger ($\frac{1}{5}$), and it will be seen that 4 is the quotient resulting from the division of one denominator (20) by the other (5).

$$\begin{array}{rccc} & \text{fifths.} & & \text{twentieths.} \\ 5) & 5 & = & 20 \\ \hline & 1 & = & 4 \end{array}$$

Let us next take the fractional units $\frac{1}{7}$ and $\frac{1}{18}$, whose denominators (7 and 18) are *not* related as measure and multiple, and we shall find that—because of this circumstance—the smaller unit ($\frac{1}{18}$) is *not* contained an exact number of times in the larger ($\frac{1}{7}$). The number of *sevenths* in an integral unit is 7, whilst the number of *eighteenths* is 18. We thus have 7 *sevenths*=18 *eighteenths*. Dividing by 7, we see that 1 *seventh* is not equal to an exact number of *eighteenths*, 7 not being contained an exact number of times in 18: in other words, 7 and 18—the two denominators—not being related as measure and multiple.

$$\begin{array}{rccc} & \text{sevenths.} & & \text{eighteenths.} \\ 7) & 7 & = & 18 \\ \hline & 1 & = & \end{array}$$

As an additional illustration, let us take the fractional units $\frac{1}{8}$ and $\frac{1}{30}$, whose denominators (8 and 30) are not related as measure and multiple, and it will be found that in this case, as in the last, the smaller unit ($\frac{1}{30}$) is not contained an exact number of times in the larger ($\frac{1}{8}$). The number of *eighths* in an integral unit is 8, whilst the number of *thirtieths* is 30. We thus have 8 *eighths*=30 *thirtieths*. Dividing by 8, we see that 1 *eighth* is not equal to an exact number of *thirtieths*, 8 not being contained an exact number of times in 30: in other words, 8 and 30—the two denominators—not being related as measure and multiple.

$$\begin{array}{rccc} & \text{eighths.} & & \text{thirtieths.} \\ 8) & 8 & = & 30 \\ \hline & 1 & = & \end{array}$$

It thus appears that, of the two fractional units $\frac{1}{2}$ and $\frac{1}{10}$, for instance, the smaller ($\frac{1}{10}$) is contained exactly 5 times in the larger ($\frac{1}{2}$)—one of the denominators (2) being contained exactly 5 times in the other (10); that, of the two fractional units $\frac{1}{3}$ and $\frac{1}{18}$, the smaller ($\frac{1}{18}$) is contained exactly 6 times in the larger ($\frac{1}{3}$)—one of the denominators (3) being contained exactly 6 times in the other (18); that, of the two fractional units $\frac{1}{4}$ and $\frac{1}{28}$, the smaller ($\frac{1}{28}$) is contained exactly 7 times in the larger ($\frac{1}{4}$)—one of the denominators (4) being contained exactly 7 times in the other (28); &c. So that a fractional unit becomes

$$\left.\begin{array}{l} \textit{twice} \\ \textit{three times} \\ \textit{four} \;\; ,, \end{array}\right\} \text{as SMALL when its denominator is} \left\{\begin{array}{l} \textit{doubled}, \\ \text{trebled}, \\ \text{quadrupled}, \end{array}\right.$$

&c. &c.

The fractional unit

$$\left.\begin{array}{l} \textit{twice} \\ \textit{three times} \\ \textit{four} \;\;\text{''} \\ \textit{five} \;\;\text{''} \\ \textit{six} \;\;\text{''} \end{array}\right\} \text{as SMALL as } \tfrac{1}{5}\text{, for example, is } \left\{\begin{array}{l} \frac{1}{5\times 2}=\frac{1}{10} \\ \frac{1}{5\times 3}=\frac{1}{15} \\ \frac{1}{5\times 4}=\frac{1}{20} \\ \frac{1}{5\times 5}=\frac{1}{25} \\ \frac{1}{5\times 6}=\frac{1}{30} \end{array}\right.$$

&c. &c.

110. As many times larger as we make the DENOMINATOR of a fraction—the numerator remaining unaltered—so many times SMALLER do we make the fraction.

This follows from § 109. The half of 1 *fifth* ($\frac{1}{5}$), for instance, being 1 *tenth* ($\frac{1}{10}$), the half of any number of *fifths* must be the same number of *tenths;* so that the half of 3 *fifths* ($\frac{3}{5}$) is 3 *tenths* ($\frac{3}{10}$). Again: the third part of 1 *seventh* ($\frac{1}{7}$) being 1 *twenty-first* ($\frac{1}{21}$), the third part of any number of *sevenths* must be the same number of *twenty-firsts;* so that the third part of 4 *sevenths* ($\frac{4}{7}$) is 4 *twenty-firsts* ($\frac{4}{21}$). In like manner, the fourth part of 1 *ninth* ($\frac{1}{9}$) being 1 *thirty-sixth* ($\frac{1}{36}$), the fourth part of any number of *ninths* must be the same number of *thirty-sixths;* so that the fourth part of 5 *ninths* ($\frac{5}{9}$) is 5 *thirty-sixths* ($\frac{5}{36}$):—

$$\tfrac{3}{5}\div 2=\tfrac{3}{10};\ \tfrac{4}{7}\div 3=\tfrac{4}{21};\ \tfrac{5}{9}\div 4=\tfrac{5}{36};\ \text{\&c.}$$

$$\textit{General formula: } \frac{x}{y}\div a=\frac{x}{y\times a}.$$

111. As many times larger as we make the NUMERATOR of a fraction—the denominator remaining unaltered—so many times LARGER do we make the fraction.

This fact requires scarcely any explanation. The double of 3 *fifths* is 6 *fifths;* the treble of 4 *sevenths,* 12 *sevenths;* the quadruple of 5 *ninths,* 20 *ninths;* &c.:—

3 *fifths* × 2 = 6 *fifths*	$\frac{3}{5}\times 2=\frac{6}{5}$
4 *sevenths* × 3 = 12 *sevenths*	$\frac{4}{7}\times 3=\frac{12}{7}$
5 *ninths* × 4 = 20 *ninths*	$\frac{5}{9}\times 4=\frac{20}{9}$
&c. &c.	&c. &c.

$$\textit{General formula: } \frac{x}{y}\times a=\frac{x\times a}{y}.$$

112. **A fraction is made neither larger nor smaller when its terms are both multipled, or both divided, by the same number.**

This follows from §§ 110 and 111. Thus, of the three fractions $\frac{3}{5}$, $\frac{3}{10}$, and $\frac{6}{10}$, the first and third are each double of the second; of the three fractions $\frac{4}{7}$, $\frac{4}{21}$, and $\frac{12}{21}$, the first and third are each 3 times as large as the second; and of the three fractions $\frac{5}{8}$, $\frac{5}{32}$, and $\frac{20}{32}$, the first and third are each 4 times as large as the second. So that, in each case, the first and third are equal:

$$\tfrac{3}{5}=\tfrac{6}{10};\ \tfrac{4}{7}=\tfrac{12}{21};\ \tfrac{5}{8}=\tfrac{20}{32};\ \&c.$$

It will be observed that, of the equal fractions $\frac{3}{5}$ and $\frac{6}{10}$, the first, when its terms are both doubled, becomes the second, whilst the second fraction, when its terms are both divided by 2, becomes the first; that, of the equal fractions $\frac{4}{7}$ and $\frac{12}{21}$, the first, when its terms are both trebled, becomes the second, whilst the second fraction, when its terms are both divided by 3, becomes the first; and that, of the equal fractions $\frac{5}{8}$ and $\frac{20}{32}$, the first, when its terms are both quadrupled, becomes the second, whilst the second fraction, when its terms are both divided by 4, becomes the first.

General formula: $\dfrac{x}{y}=\dfrac{x\times a}{y\times a}$.

113. **When its terms are prime to one another, a fraction is said to be in its *simplest form* or *lowest terms*.**

The fraction $\frac{3}{5}$, for example, is in its "simplest form"—the terms (3 and 5) being prime to one another; but the fraction $\frac{24}{36}$ is not in its simplest form—the terms (24 and 36) not being prime to each other.

114. **To reduce to its simplest form a fraction whose terms are not prime to one another, we *divide the terms by their greatest common measure.****

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* If we take the fraction $\dfrac{x}{y}$, and suppose the greatest common measure of its terms to be a, we shall find that the quotients resulting from the division of x and y by a are prime to one another.

Let a be contained x' times in x, and y' times in y; and let x' and y' be not prime to each other, if possible—in other words, let x' and y' have a common measure, m. Let m be contained x'' times in x', and y'' times in y'. Then (II.), $x'=m\times x''$, and $y'=m\times y''$; (I.) $x=a\times x'$, and $y=a\times y'$: *i.e.*, ($m\times x''$ being substituted for x', and $m\times y''$ for y',) $x=a\times m\times x''$, and $y=a\times m\times y''$. So

$$\text{(I.)}\quad a\overline{)\,x\quad y\,}\atop{\;\;x'\quad y'}\qquad\qquad \text{(II.)}\quad m\overline{)\,x'\quad y'\,}\atop{\;\;x''\quad y''}$$

Thus, the fraction $\frac{24}{36}$, when its terms are divided by thei[r] greatest common measure, 12, is reduced to its simplest form $\frac{24}{36}=\frac{24\div 12}{36\div 12}=\frac{2}{3}$.

115. A fraction is said to be PROPER or IMPROPE[R] according as its numerator is, or is not, less than it[s] denominator. In other words, a fraction is calle[d] PROPER whose numerator is less than its denominator; whilst a fraction is termed IMPROPER whos[e] numerator is either equal to, or greater than, its denominator.

The following are "proper" fractions: $\frac{2}{3}$, $\frac{3}{5}$, $\frac{4}{7}$, $\frac{5}{8}$, &c. Here on the other hand, are "improper" fractions: $\frac{3}{3}$, $\frac{5}{5}$, $\frac{7}{4}$, $\frac{10}{2}$, &c.

116. A number which is partly integral and partly fractional is called a MIXED NUMBER.

The following are "mixed" numbers: $3\frac{1}{2}$, $5\frac{1}{3}$, $2\frac{3}{4}$, $4\frac{7}{8}$, &c.

117. To convert a mixed number into an imprope[r] fraction, we *multiply the integral part of the mixed number by the denominator of the fractional part, and add the numerator of the fractional part to the product; set down the result for numerator, and the denominator of the fractional part for denominator.*

Thus: $3\frac{1}{2}=\frac{3\times 2+1}{2}=\frac{6+1}{2}=\frac{7}{2}$; $5\frac{1}{3}=\frac{5\times 3+1}{3}=\frac{15+1}{3}=\frac{16}{3}$; $2\frac{3}{4}=\frac{2\times 4+3}{4}=\frac{8+3}{4}=\frac{11}{4}$; &c. This is easily understood. We convert a whole number into *halves* by multiplying by 2, into *thirds* by multiplying by 3, into *fourths* by multiplying by 4, &c. A yard of ribbon, for example, being divisible into 2 *halves*, a length of 3 yards is divisible into ($3\times 2=$) 6 *halves* of a yard; an acre of land being divisible into 3 *thirds*, a field containing 5 acres is divisible into ($5\times 3=$) 15 *thirds* of an acre; a pound of butter being divisible into 4 *fourths*, a roll

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that $a\times m$ is a common measure of x and y. But this is impossible, m being the *greatest* common measure of x and y. It follows, therefore, that x' and y' *are* prime to one another.

weighing 2 pounds is divisible into $(2\times4=)$ 8 *fourths* of a pound; &c. We thus have—

$3\frac{1}{2}=3+\frac{1}{2}=6$ *halves* $+1$ *half* $=7$ *halves* $=\frac{7}{2}$;
$5\frac{1}{3}=5+\frac{1}{3}=15$ *thirds* $+1$ *third* $=16$ *thirds* $=\frac{16}{3}$;
$2\frac{3}{4}=2+\frac{3}{4}=8$ *fourths* $+3$ *fourths* $=11$ *fourths* $=\frac{11}{4}$; &c.

118. To convert an improper fraction into a whole number or mixed number—as the case may be,* we *divide the numerator by the denominator, and set down the quotient as the integral part of the required result. When there is a remainder, we write the denominator under it, and set down the fraction so obtained as the fractional part of the answer.*

Thus, 10 *halves* of a penny are equivalent to $(10\div2=)$ 5 pence—the number of *halves* in a penny being 2; 18 *thirds* of a shilling are equivalent to $(18\div3=)$ 6 shillings—the number of *thirds* in a shilling being 3; 31 *fourths* of a yard are equivalent to $(31\div4=)$ 7 yards and 3 *fourths* of a yard—the number of *fourths* in a yard being 4; 41 *fifths* of an acre are equivalent to $(41\div5=)$ 8 acres and 1 *fifth* of an acre—the number of *fifths* in an acre being 5; and so on:

$$\frac{10}{2}=5;\ \frac{18}{3}=6;\ \frac{31}{4}=7\tfrac{3}{4};\ \frac{41}{5}=8\tfrac{1}{5};\ \&c.$$

119. When we multiply a fraction by its denominator, we obtain the numerator for product.

This follows from §§ 111 and 118:
$\frac{2}{3}\times3=(\frac{6}{3}=)2$; $\frac{3}{5}\times5=(\frac{15}{5}=)3$; $\frac{4}{7}\times7=(\frac{28}{7}=)4$; $\frac{5}{9}\times9=(\frac{45}{9}=)5$; &c.

$$\textit{General formula: } \frac{x}{y}\times y=\left(\frac{x\times y}{y}=\right)x.$$

120. A fractional NUMBER can be read in either of two ways. For instance:

$\frac{2}{3}$ can be read "2 *thirds* of 1," or "1 *third* of 2;"
$\frac{3}{5}$,, "3 *fifths* of 1," or "1 *fifth* of 3;"
$\frac{4}{7}$,, "4 *sevenths* of 1," or "1 *seventh* of 4;"
$\frac{5}{9}$,, "5 *ninths* of 1," or "1 *ninth* of 5;" &c.

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* According as the numerator is, or is not, a multiple of the denominator.

This follows from § 119, as we shall find on reflecting that, when the product of two factors is divided by one of them, the other factor is obtained for quotient. Thus, 2 being the product of $\frac{2}{3}$ by 3, if we divide 2 by 3—that is, take a *third* of 2—we must obtain $\frac{2}{3}$; 3 being the product of $\frac{3}{5}$ by 5, if we divide 3 by 5—that is, take a *fifth* of 3—we must obtain $\frac{3}{5}$; 4 being the product of $\frac{4}{7}$ by 7 (in other words, 4 being 7 times $\frac{4}{7}$), $\frac{4}{7}$ must be the *seventh* part of 4; 5 being the product of $\frac{5}{9}$ by 9 (in other words, 5 being 9 times $\frac{5}{9}$), $\frac{5}{9}$ must be the *ninth* part of 5; &c.:

$\frac{2}{3}\times 3=2$	$2\div 3$ (*i.e.*, $\frac{1}{3}$ of 2)$=\frac{2}{3}$
$\frac{3}{5}\times 5=3$	$3\div 5$ („ $\frac{1}{5}$ of 3)$=\frac{3}{5}$
$\frac{4}{7}\times 7=4$	$4\div 7$ („ $\frac{1}{7}$ of 4)$=\frac{4}{7}$
$\frac{5}{9}\times 9=5$	$5\div 9$ („ $\frac{1}{9}$ of 5)$=\frac{5}{9}$
&c.	&c.

The fact could also be established in this way:

2=6 *thirds*; $\frac{1}{3}$ of 2=$\frac{1}{3}$ of 6 *thirds*=2 *thirds*=$\frac{2}{3}$
3=15 *fifths*; $\frac{1}{5}$ of 3=$\frac{1}{5}$ of 15 *fifths*=3 *fifths*=$\frac{3}{5}$
4=28 *sevenths*; $\frac{1}{7}$ of 4=$\frac{1}{7}$ of 28 *sevenths*=4 *sevenths*=$\frac{4}{7}$
5=45 *ninths*; $\frac{1}{9}$ of 5=$\frac{1}{9}$ of 45 *ninths*=5 *ninths*=$\frac{5}{9}$

121. To write an integer under the form of a fraction, we *set down the integer for numerator, and* 1 *for denominator.*

Thus, $3=\frac{3}{1}$; $5=\frac{5}{1}$; $8=\frac{8}{1}$; &c.

This is explained by the fact that (§ 120) a fraction may be regarded as an expression indicating that the numerator is to be divided by the denominator.

122. We convert a fraction of a shilling into a fraction of a penny, or a fraction of a yard into a fraction of a foot, or a fraction of a pennyweight into a fraction of a grain, &c., when—whilst retaining the denominator—we multiply the numerator by the number of pence in a shilling, or the number of feet in a yard, or the number of grains in a pennyweight, &c.—as the case may be.

Thus, $\frac{3}{5}s.=\left(\frac{3\times 12}{5}=\right)\frac{36}{5}d.$; $\frac{4}{7}\ yd.=\left(\frac{4\times 3}{7}=\right)\frac{12}{7}\ ft.$; $\frac{5}{11}\ dwt.=\left(\frac{5\times 24}{11}=\right)\frac{120}{11}\ gr.$; &c. Because (§ 120) 3 *fifths* of 1*s.* are equivalent to 1 *fifth* of 3*s.*, or to 1 *fifth* of (3 × 12=) 36*d.*;

and 1 *fifth* of 36*d.* is equivalent to 36 *fifths* ($\frac{36}{5}$) of 1*d.*: 4 *sevenths* of 1 *yd.* are equivalent to 1 *seventh* of 4 *yds.*, or to 1 *seventh* of $(4\times3=)12$ *ft.*; and 1 *seventh* of 12 *ft.* is equivalent to 12 *sevenths* ($\frac{12}{7}$) of 1 *ft.*: 5 *elevenths* of 1 *dwt.* are equivalent to 1 *eleventh* of 5 *dwts.*, or to 1 *eleventh* of $(5\times24=)120$ *grs.*; and 1 *eleventh* of 120 *grs.* is equivalent to 120 *elevenths* ($\frac{120}{11}$) of 1 *gr.*: &c.

123. *Reduction of Fractions to a "Common" Denominator.*—By this is meant, the reduction of fractions to equivalent ones—all having the *same* denominator. Such reduction is necessary whenever, in the case of either Fractional Addition or Fractional Subtraction, the fractions to be dealt with have not a common denominator.

Let it be required to reduce $\frac{2}{3}$, $\frac{3}{4}$, and $\frac{5}{6}$ to a common denominator. The first of these fractions ($\frac{2}{3}$) is composed of *thirds*, and the denominations to which *thirds* can be reduced are—*sixths*, *ninths*, *twelfths*, *fifteenths*, *eighteenths*, &c.; a *third* being divisible (§ 109) into an exact number (2) of *sixths*, or an exact number (3) of *ninths*, or an exact number (4) of *twelfths*, &c. But *thirds* could not be reduced to any of the intermediate denominations, such as *fourths*, *sevenths*, *elevenths*, &c.: because a *third* does not contain an exact number of *fourths*, or of *sevenths*, or of *elevenths*, &c. So that any fraction to which $\frac{2}{3}$ can be reduced must have 6, or 9, or 12, &c. for denominator: in other words, must have for denominator a MULTIPLE of the given denominator, 3.

The second of the given fractions ($\frac{3}{4}$) is composed of *fourths*, and the denominations to which *fourths* can be reduced are—*eighths*, *twelfths*, *sixteenths*, *twentieths*, *twenty-fourths*, &c.; a *fourth* being divisible into an exact number (2) of *eighths*, or an exact number (3) of *twelfths*, or an exact number (4) of *sixteenths*, &c. But *fourths* could not be reduced to any of the intermediate denominations, such as *sixths*, *tenths*, *fifteenths*, &c.: because a *fourth* does not contain an exact number of *sixths*, or of *tenths*, or of *fifteenths*, &c. So that any fraction to which $\frac{3}{4}$ can be reduced must have 8, or 12, or 16, &c. for denominator: in other words, must have for denominator a MULTIPLE of the given denominator, 4.

The third of the given fractions ($\frac{5}{6}$) is composed of *sixths*, and the denominations to which *sixths* can be reduced are—*twelfths*, *eighteenths*, *twenty-fourths*, *thirtieths*, *thirty-sixths*, &c.; a *sixth* being divisible into an exact number (2) of *twelfths*, or an exact number (3) of *eighteenths*, or an exact

number (4) of *twenty-fourths*, &c. But *sixths* could not be reduced to any of the intermediate denominations, such as *ninths*, *fourteenths*, *twentieths*, &c.: because a *sixth* does not contain an exact number of *ninths*, or of *fourteenths*, or of *twentieths*, &c. So that any fraction to which $\frac{5}{6}$ can be reduced must have 12, or 18, or 24, &c. for denominator: in other words, must have for denominator a MULTIPLE of the given denominator, 6.

In order, therefore, to reduce $\frac{2}{3}$, $\frac{3}{4}$, and $\frac{5}{6}$ to equivalent fractions having the same denominator, we must begin by selecting, for "common denominator," a number which is a multiple of 3 (the first fraction's denominator), and of 4 (the second fraction's denominator), and of 6 (the third fraction's denominator)—that is, a number which is a *common* multiple of 3, 4, and 6. Of the innumerable common multiples which there are of 3, 4, and 6, we naturally select the *least*, 12; and, to reduce the given fractions to equivalent ones having 12 for denominator, we multiply the terms of the first ($\frac{2}{3}$) by 4, the terms of the second ($\frac{3}{4}$) by 3, and the terms of the third ($\frac{5}{6}$) by 2—the multiplier, it will be perceived, being in each case the number indicating how often the denominator is contained in 12. We then have $\frac{2}{3}$, $\frac{3}{4}$, and $\frac{5}{6}$ under the forms $\frac{8}{12}$, $\frac{9}{12}$, and $\frac{10}{12}$, respectively.

$$\frac{2}{3}=\frac{2\times4}{3\times4}=\frac{8}{12}$$

$$\frac{3}{4}=\frac{3\times3}{4\times3}=\frac{9}{12}$$

$$\frac{5}{6}=\frac{5\times2}{6\times2}=\frac{10}{12}$$

Next, let it be required to reduce $\frac{7}{8}$, $\frac{4}{9}$, $\frac{3}{10}$, and $\frac{5}{12}$ to a common denominator. The least common multiple (obtained as shown in the margin) of the denominators is 360, in which the first denominator (8) is contained 45 times: we therefore multiply the terms of the first fraction ($\frac{7}{8}$) by 45. The second denominator (9) is contained 40 times in 360: we therefore multiply the terms of the second fraction ($\frac{4}{9}$) by 40. The third denominator (10) being contained 36 times in 360, we multiply the terms of the third fraction ($\frac{3}{10}$) by 36. And, the fourth denominator (12) being contained 30 times in 360, we multiply the terms of the fourth fraction ($\frac{5}{12}$) by 30. The given fractions then appear under the forms $\frac{315}{360}$, $\frac{160}{360}$, $\frac{108}{360}$, and $\frac{150}{360}$, respectively.

$$\begin{array}{r|rrrr} 2) & 8 & 9 & 10 & 12 \\ \hline 2) & 4 & 9 & 5 & 6 \\ \hline & 2 & 9 & 5 & 3 \end{array}$$

L.C.M. =

$$2\times2\times2\times9\times5=360$$

$$\frac{7}{8}=\frac{7\times45}{8\times45}=\frac{315}{360}$$

$$\frac{4}{9}=\frac{4\times40}{9\times40}=\frac{160}{360}$$

$$\frac{3}{10}=\frac{3\times36}{10\times36}=\frac{108}{360}$$

$$\frac{5}{12}=\frac{5\times30}{12\times30}=\frac{150}{360}$$

124. To reduce fractions to a common denominator, we *find the least common multiple of the denominators, and multiply the terms of each fraction by the number indicating how often the denominator is contained in the least common multiple.*

NOTE.—When the denominators happen to be all prime to one another, their continued product is the "least common multiple:" in such cases, therefore, the terms of each fraction are multiplied by the product of the denominators of the *other* fractions.

Taking, for example $\frac{3}{5}$, $\frac{4}{7}$, and $\frac{8}{9}$, we see that the least common multiple of the denominators, which are prime to one another, is $5\times7\times9$, and that in this least common multiple the first denominator (5) is contained 7×9 times; the second denominator (7), 5×9 times; and the third denominator (9), 5×7 times. In order, therefore, to reduce $\frac{3}{5}$, $\frac{4}{7}$, and $\frac{8}{9}$ to a common denominator, we multiply the terms of the first fraction ($\frac{3}{5}$) by 7×9, those of the second fraction ($\frac{4}{7}$) by 5×9, and those of the third fraction ($\frac{8}{9}$) by 5×7: that is, we multiply the terms of each fraction by the product of the denominators of the other fractions.

$$\frac{3}{5}=\frac{3\times7\times9}{5\times7\times9}=\frac{189}{315}$$

$$\frac{4}{7}=\frac{4\times5\times9}{7\times5\times9}=\frac{180}{315}$$

$$\frac{8}{9}=\frac{8\times5\times7}{9\times5\times7}=\frac{280}{315}$$

FRACTIONAL ADDITION.

125. Addition is called FRACTIONAL when the addends are fractional numbers (of the *same kind*).

EXAMPLE I.—Find the sum of $\frac{3}{7}$ and $\frac{2}{7}$; of $\frac{7}{9}$, $\frac{5}{9}$, and $\frac{4}{9}$; and of $\frac{8}{11}$, $\frac{6}{11}$, $\frac{5}{11}$, and $\frac{3}{11}$.

It is obvious that the sum of 3 *sevenths* and 2 *sevenths* is 5 *sevenths*; that the sum of 7 *ninths*, 5 *ninths*, and 4 *ninths* is 16 *ninths*; and that the sum of 8 *elevenths*, 6 *elevenths*, 5 *elevenths*, and 3 *elevenths* is 22 *elevenths*:

$$\frac{3}{7}+\frac{2}{7}=\frac{5}{7}\,;\ \frac{7}{9}+\frac{5}{9}+\frac{4}{9}=\frac{16}{9}=1\tfrac{7}{9}\,;\ \frac{8}{11}+\frac{6}{11}+\frac{5}{11}+\frac{3}{11}=\frac{22}{11}=2.$$

126. Rule for Fractional Addition, when the

addends have a common denominator: Set dc the sum of the numerators for numerator, and common denominator for denominator.

EXAMPLE II.—Find the sum of $\frac{7}{8}$, $\frac{4}{9}$, $\frac{3}{10}$, and $\frac{5}{12}$.

Reducing these fractions to a common denominator, we l (see p. 132)—

$$\tfrac{7}{8}+\tfrac{4}{9}+\tfrac{3}{10}+\tfrac{5}{12}=\tfrac{315}{360}+\tfrac{160}{360}+\tfrac{108}{360}+\tfrac{150}{360}=\tfrac{733}{360}=2\tfrac{13}{360}.$$

127. Rule for Fractional Addition, when addends have not a common denominator: Red the addends to a common denominator, and tl proceed as already directed (§ 126).

NOTE 1.—If it were required to find the sum of (say) a *yard*, $\frac{5}{8}$ of a *foot*, and $\frac{3}{4}$ of an *inch*, we should begin by c verting the first two fractions into fractions of an *inch*: $\frac{2}{5}$ *yard* into $\left(\frac{2\times36}{5}=\right)\frac{72}{5}$ of an *inch*, and $\frac{5}{8}$ of a *foot* into $\left(\frac{5\times12}{8}=\right.$ or $\frac{15}{2}$ of an *inch* (see § 122). The work would then ass this form:

$\frac{72}{5}$ *in.* $+\frac{15}{2}$ *in.* $+\frac{3}{4}$ *in.* $=\frac{288}{20}$ *in.* $+\frac{150}{20}$ *in.* $+\frac{15}{20}$ *in.* $=\frac{453}{20}$ *in.* $=22\frac{1}{2}$

NOTE 2.—The addition of MIXED numbers involves Simpl well as Fractional Addition. Let it be required, for insta to find the sum of $8\frac{2}{3}$, $7\frac{3}{4}$, and $9\frac{5}{6}$. By means of Fracti Addition we find the sum of the fractional portions, $\frac{2}{3}$ and $\frac{5}{6}$, to be (see p. 132) $\frac{8}{12}+\frac{9}{12}+\frac{10}{12}=\frac{27}{12}$; whilst, by mean Simple Addition, we find the sum of the integral portions, 8 and 9, to be 24. The required sum, therefore, is $24\frac{27}{12}$: $+7\frac{3}{4}+9\frac{5}{6}=8+7+9+\frac{2}{3}+\frac{3}{4}+\frac{5}{6}=24+\frac{27}{12}=24\frac{27}{12}$. Under o nary circumstances, however, $\frac{27}{12}$ would be converted into mixed number $(2\frac{3}{12}=)2\frac{1}{4}$, and $24\frac{27}{12}$ written under the f $(24+2\frac{1}{4}=)\ 26\frac{1}{4}$.

FRACTIONAL SUBTRACTION.

128. Subtraction is called FRACTIONAL when t numbers whose difference is required (and wh must be of the *same kind*) are fractional, or wh only one of them is a fractional number.

EXAMPLE I.—Find the difference between $\frac{9}{13}$ and $\frac{4}{13}$; tween $\frac{15}{17}$ and $\frac{3}{17}$; and between $\frac{18}{19}$ and $\frac{5}{19}$.

If 4 *thirteenths* were taken from 9 *thirteenths*, 5 *thirteenths* would be left; if 3 *seventeenths* were taken from 15 *seventeenths*, 12 *seventeenths* would be left; and if 5 *nineteenths* were taken from 18 *nineteenths*, 13 *nineteenths* would be left:—

$$\tfrac{9}{13}-\tfrac{4}{13}=\tfrac{5}{13};\ \tfrac{15}{17}-\tfrac{3}{17}=\tfrac{12}{17};\ \tfrac{18}{19}-\tfrac{5}{19}=\tfrac{13}{19}.$$

129. Rule for Fractional Subtraction, when the fractions have a common denominator: Set down the difference of the numerators for numerator, and the common denominator for denominator.

EXAMPLE II.—Find the difference between $\frac{7}{8}$ and $\frac{5}{6}$.

These fractions must first be reduced to a common denominator. The least common multiple of the denominators is 24. in which 8 and 6 are contained 3 times and 4 times, respectively. We therefore multiply the terms of the first fraction ($\frac{7}{8}$) by 3, and those of the second ($\frac{5}{6}$) by 4. We then have $\frac{7}{8}=\frac{7\times3}{8\times3}=\frac{21}{24}$; $\frac{5}{6}=\frac{5\times4}{6\times4}=\frac{20}{24}$; $\frac{7}{8}-\frac{5}{6}=\frac{21}{24}-\frac{20}{24}=\frac{1}{24}$.

130. Rule for Fractional Subtraction, when the fractions have not a common denominator: Reduce the fractions to a common denominator, and then proceed as already directed (§ 129).

NOTE 1.—In finding the difference between (say) $8\frac{3}{4}$ and $2\frac{5}{6}$ we begin by converting each of the mixed numbers into an improper fraction (§ 117). And in finding the difference between 9 and $3\frac{5}{7}$, we begin by writing 9 under the form $\frac{9}{1}$ (§ 121), and $3\frac{5}{7}$ under the form $\frac{26}{7}$. The work is shown in the margin.

$8\frac{3}{4}-2\frac{5}{6}=$	$9-3\frac{5}{7}=$
$\frac{35}{4}-\frac{17}{6}=$	$\frac{9}{1}-\frac{26}{7}=$
$\frac{105}{12}-\frac{34}{12}=$	$\frac{63}{7}-\frac{26}{7}=$
$\frac{71}{12}$ or $5\frac{11}{12}$	$\frac{37}{7}$ or $5\frac{2}{7}$

NOTE 2.—In finding the difference between $\frac{2}{3}$ of a *gallon* and $\frac{3}{4}$ of a *pint*, we begin by converting the first fraction into $\left(\frac{2\times8}{3}=\right)$ $\frac{16}{3}$ of a *pint*. The required difference is then found as shown in the margin.

$\frac{2}{3}$ gal. $-\frac{3}{4}$ pt. $=$
$\frac{16}{3}$ pt. $-\frac{3}{4}$ pt. $=$
$\frac{64}{12}$ pt. $-\frac{9}{12}$ pt. $=$
$\frac{55}{12}$ pt. or $4\frac{7}{12}$ pts.

FRACTIONAL MULTIPLICATION.

131. Multiplication is called FRACTIONAL when the factors are fractional numbers, or when only one of them is fractional.

EXAMPLE I.—Multiply $\frac{4}{7}$ by $\frac{3}{5}$.

The product of $\frac{4}{7}$ by 3 (§ 111) being $\frac{12}{7}$, the product of $\frac{4}{7}$ by the *fifth* part of 3, or by $\frac{3}{5}$, must be the *fifth* part of $\frac{12}{7}$—that is, $\frac{12}{7} \div 5 = \frac{12}{35}$ (§ 110).

EXAMPLE II.—Multiply $\frac{5}{9}$ by $\frac{4}{7}$.

The product of $\frac{5}{9}$ by 4 being $\frac{20}{9}$, the product of $\frac{5}{9}$ by the *seventh* part of 4, or by $\frac{4}{7}$, must be the *seventh* part of $\frac{20}{9}$—that is, $\frac{20}{9} \div 7 = \frac{20}{63}$.

EXAMPLE III.—Multiply $\frac{7}{8}$ by $\frac{5}{9}$.

The product of $\frac{7}{8}$ by 5 being $\frac{35}{8}$, the product of $\frac{7}{8}$ by the *ninth* part of 5, or by $\frac{5}{9}$, must be the *ninth* part of $\frac{35}{8}$—that is, $\frac{35}{8} \div 9 = \frac{35}{72}$:

$$\frac{4}{7}\times\frac{3}{5}=\frac{4\times 3}{7\times 5};\ \frac{5}{9}\times\frac{4}{7}=\frac{5\times 4}{9\times 7};\ \frac{7}{8}\times\frac{5}{9}=\frac{7\times 5}{8\times 9};\ \text{\&c.}$$

General formula: $\dfrac{x}{y}\times\dfrac{a}{b}=\dfrac{x\times a}{y\times b}$

132. Rule for Fractional Multiplication: Set down the product of the numerators for numerator, and the product of the denominators for denominator.

NOTE 1.—When a mixed number occurs amongst the factors, we begin by converting it into an improper fraction (§ 117).

Thus, $5\frac{2}{3}\times\frac{7}{8}=\frac{17}{3}\times\frac{7}{8}=\frac{17\times 7}{3\times 8}=\frac{119}{24}=4\frac{23}{24}$.

NOTE 2.—When one of the factors is an integer, it is usual to write it under the form of a fraction (§ 121)—particularly when there are more than two factors. Thus, $\frac{3}{7}\times 2=\frac{3}{7}\times\frac{2}{1}=\frac{6}{7}$; $\frac{3}{5}\times\frac{2}{11}\times 7=\frac{3}{5}\times\frac{2}{11}\times\frac{7}{1}=\frac{42}{55}$; &c. So that the last Rule applies to the multiplication of a fraction by an integer, as well as to the multiplication of one fraction by another.

FRACTIONAL DIVISION.

133. Division is called FRACTIONAL when the dividend, the divisor, and the resulting quotient are fractional numbers; or when any one of them is fractional.

EXAMPLE I.—Divide $\frac{4}{7}$ by $\frac{3}{5}$.

The division of $\frac{4}{7}$ by 3 gives $\frac{4}{21}$ for quotient (§ 110); therefore the division of $\frac{4}{7}$ by the *fifth* part of 3, or by $\frac{3}{5}$, must give 5 times as large a quotient—that is, $\frac{4}{21}\times 5=\frac{20}{21}$ (§ 111).*

EXAMPLE II.—Divide $\frac{5}{9}$ by $\frac{4}{7}$.

The division of $\frac{5}{9}$ by 4 gives $\frac{5}{36}$ for quotient; therefore the division of $\frac{5}{9}$ by the *seventh* part of 4, or by $\frac{4}{7}$, must give 7 times as large a quotient—that is, $\frac{5}{36}\times 7=\frac{35}{36}$.

EXAMPLE III.—Divide $\frac{7}{8}$ by $\frac{5}{9}$.

The division of $\frac{7}{8}$ by 5 gives $\frac{7}{40}$ for quotient; therefore the division of $\frac{7}{8}$ by the *ninth* part of 5, or by $\frac{5}{9}$, must give 9 times as large a quotient—that is, $\frac{7}{40}\times 9=\frac{63}{40}$.

From the preceding examples it appears that, to divide by $\frac{3}{5}$, we multiply by 5 and divide by 3; that, to divide by $\frac{4}{7}$, we multiply by 7 and divide by 4; and that, to divide by $\frac{5}{9}$, we multiply by 9 and divide by 5. So that dividing by $\frac{3}{5}$ is the same as multiplying by $\frac{5}{3}$; dividing by $\frac{4}{7}$, the same as multiplying by $\frac{7}{4}$; dividing by $\frac{5}{9}$, the same as multiplying by $\frac{9}{5}$; &c.:—

$$\frac{4}{7}\div\frac{3}{5}=\frac{4}{7}\times\frac{5}{3}\,;\ \frac{5}{9}\div\frac{4}{7}=\frac{5}{9}\times\frac{7}{4}\,;\ \frac{7}{8}\div\frac{5}{9}=\frac{7}{8}\times\frac{9}{5}\,;\ \&c.$$

General formula: $\frac{x}{y}\div\frac{a}{b}=\frac{x}{y}\times\frac{b}{a}$.

134. Rule for Fractional Division: Invert the divisor, and then treat the exercise as one in Fractional Multiplication (§ 132).

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* Whenever we divide the "true" dividend by too large a divisor, we obtain a quotient as many times too *small* as the divisor is too large. Thus, dividing 36 by 3, we obtain 12 for quotient; dividing 36 by the double of 3, we obtain twice as small a quotient; dividing 36 by the treble of 3, we obtain a quotient three times as small as the first quotient; and so on:—

$$36\div 3=12\,;\ 36\div 6=6\,;\ 36\div 9=4\,;\ 36\div 12=3\,;\ \&c.$$

NOTE 1.—A fraction is said to be "inverted" when its terms are interchanged: the denominator becoming numerator, and the numerator becoming denominator.

NOTE 2.—This Rule applies to the division of a fraction by an integer. Thus, $\frac{7}{8} \div 5 = \frac{7}{8} \div \frac{5}{1} = \frac{7}{8} \times \frac{1}{5} = \frac{7}{40}$.

NOTE 3.—When either the dividend or the divisor is a mixed number, we begin by converting it into an improper fraction (§ 117). For instance: $6\frac{3}{4} \div \frac{5}{7} = \frac{27}{4} \div \frac{5}{7} = \frac{27}{4} \times \frac{7}{5} = \frac{189}{20} = 9\frac{9}{20}$.

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135. A fraction whose terms are integers is called a SIMPLE fraction.

So that all the fractions—proper and improper—which we have just been considering, are "simple" fractions: each having a whole number for numerator, and a whole number for denominator. A fraction which is not "simple" is either *compound* or *complex*.

136. By a COMPOUND fraction is meant—a fraction of a fraction.

The following are "compound" fractions: $\frac{3}{4}$ of $\frac{5}{7}$; $\frac{2}{3}$ of $\frac{4}{5}$ of $\frac{7}{11}$; &c.

137. To convert a COMPOUND fraction into a SIMPLE fraction, we *substitute the sign of multiplication* (×), *or conceive it substituted, for the word "of," and then perform the operation (in Fractional Multiplication) so indicated.*

Thus, $\frac{3}{4}$ of $\frac{5}{7} = \frac{3}{4} \times \frac{5}{7} = \frac{15}{28}$; $\frac{2}{3}$ of $\frac{4}{5}$ of $\frac{7}{11} = \frac{2}{3} \times \frac{4}{5} \times \frac{7}{11} = \frac{56}{165}$; &c. This is easily explained: 1 *fourth* ($\frac{1}{4}$) of $\frac{5}{7}$ being ($\frac{5}{7} \div 4 =$) $\frac{5}{28}$, 3 *fourths* ($\frac{3}{4}$) of $\frac{5}{7}$ must be 3 times $\frac{5}{28}$—that is, ($\frac{5}{28} \times 3 =$) $\frac{15}{28}$; so that $\frac{3}{4}$ of $\frac{5}{7} = \frac{3}{4} \times \frac{5}{7}$. Again: 1 *fifth* ($\frac{1}{5}$) of $\frac{7}{11}$ being ($\frac{7}{11} \div 5 =$) $\frac{7}{55}$, 4 *fifths* ($\frac{4}{5}$) of $\frac{7}{11}$ must be 4 times $\frac{7}{55}$—that is, ($\frac{7}{55} \times 4 =$) $\frac{28}{55}$; 1 *third* ($\frac{1}{3}$) of $\frac{28}{55}$ being ($\frac{28}{55} \div 3 =$) $\frac{28}{165}$, 2 *thirds* ($\frac{2}{3}$) of $\frac{28}{55}$ must be the double of $\frac{28}{165}$—that is, ($\frac{28}{165} \times 2 =$) $\frac{56}{165}$; so that $\frac{2}{3}$ of $\frac{4}{5}$ of $\frac{7}{11} = \frac{2}{3}$ of $\frac{4}{5} \times \frac{7}{11} = \frac{2}{3} \times \frac{4}{5} \times \frac{7}{11}$. A "compound" fraction, therefore, might be regarded as an unworked exercise in Fractional Multiplication.

138. A fraction whose terms are not both integers is called a COMPLEX fraction.

The following are "complex" fractions—each of them having either a fraction or a mixed number for numerator, or for denominator, or for both:

$$\frac{\frac{2}{3}}{4} \quad \frac{2}{\frac{3}{4}} \quad \frac{\frac{7}{8}}{\frac{5}{6}} \quad \frac{3\frac{1}{2}}{\frac{4}{5}} \quad \frac{6}{2\frac{1}{4}} \quad \frac{8\frac{2}{3}}{6\frac{3}{5}}$$

139. To convert a COMPLEX fraction into a SIMPLE fraction, we *divide the numerator by the denominator.*

Thus, $\frac{\frac{2}{3}}{4}=\frac{2}{3}\div 4=\frac{2}{12}=\frac{1}{6}$; $\frac{2}{\frac{3}{4}}=2\div\frac{3}{4}=2\times\frac{4}{3}=\frac{8}{3}=2\frac{2}{3}$; $\frac{\frac{7}{8}}{\frac{5}{6}}=\frac{7}{8}\div\frac{5}{6}$ $=\frac{7}{8}\times\frac{6}{5}=\frac{42}{40}=\frac{21}{20}=1\frac{1}{20}$; $\frac{3\frac{1}{2}}{\frac{4}{5}}=3\frac{1}{2}\div\frac{4}{5}=\frac{7}{2}\times\frac{5}{4}=\frac{35}{8}=4\frac{3}{8}$; &c.

NOTE 1.—As a "simple" fraction might be considered an unworked exercise in Simple Division, ($\frac{5}{8}$, for example, indicating that the *eighth* part of 5 is to be taken, or that 5 is to be divided by 8)—so, a "complex" fraction might be looked upon as an unworked exercise in Fractional Division.

NOTE 2.—The line employed to separate the terms of a complex fraction is *longer* and *thicker* than the line by which the terms of a simple fraction are separated. Because, otherwise, such complex fractions as, for example, $\frac{\frac{2}{3}}{4}$ and $\frac{2}{\frac{3}{4}}$ which—as we have seen—are very different, would be liable to be confounded with each other.

NOTE 3.—The word "fraction" is always understood to mean "SIMPLE fraction" when nothing to the contrary is mentioned.

140. A fraction—whether proper or improper—which has either 10 or a power of 10 for denominator, is called a DECIMAL fraction.

The following are "decimal" fractions : $\frac{3}{10}$, $\frac{789}{100}$, $\frac{67}{1000}$, $\frac{54321}{10000}$, &c.

141. To convert a DECIMAL into a decimal FRACTION, we *set down, for numerator, what the decimal would represent if it were a whole number; and, for denominator, write as many ciphers after the digit* 1 *as there are figures in the decimal.*

Thus, $\cdot7=\cdot7\times\frac{10}{10}=\frac{\cdot7\times10}{10}=\frac{7}{10}$; $\cdot89=\cdot89\times\frac{100}{100}=\frac{\cdot89\times100}{100}=\frac{89}{100}$; $\cdot043=\cdot043\times\frac{1000}{1000}=\frac{\cdot043\times1000}{1000}=\frac{43}{1000}$; &c.

142. A fraction whose denominator is neither 10 nor a power of 10 is called a VULGAR fraction.

The following are "vulgar" fractions: $\frac{5}{8}$, $\frac{7}{3}$, $\frac{6}{11}$, $\frac{30}{5}$, &c. So that every fraction is either "vulgar" or "decimal."

As "complex" fractions can be converted into "simple" fractions by means of Fractional Division (§ 139)—so, simple fractions can be converted into simple numbers by means of Simple Division.

143. To convert a DECIMAL fraction into a simple number, we *set down the numerator* (by itself), *and remove the decimal point as many places to the left as there are ciphers in the denominator.*

Thus, $\frac{3}{10}=3\div10=\cdot3$; $\frac{789}{100}=789\div100=7\cdot89$; $\frac{67}{1000}=67\div 1{,}000=\cdot067$; &c.

NOTE.—The Fractional Rules are employed less frequently in the case of decimal fractions than in that of vulgar fractions, because of the facility with which decimal fractions can be converted into—and treated as—simple numbers As an illustration, let it be required to find (*a*) $\frac{9}{10}+\frac{321}{100}+\frac{47}{1000}$; (*b*) $\frac{53}{100}-\frac{79}{1000}$; (*c*) $\frac{549}{1000}\times\frac{83}{100}$; and (*d*) $\frac{45567}{100000}\div\frac{83}{100}$.

(*a*.) By means of Fractional Addition we find $\frac{9}{10}+\frac{321}{100}+\frac{47}{1000}=\frac{900}{1000}+\frac{3210}{1000}+\frac{47}{1000}=\frac{4157}{1000}$ or $(4{,}157\div1{,}000=)\ 4\cdot157$; whilst by means of Simple Addition [$\frac{9}{10}$ being converted into $(9\div10=)\ \cdot9$, $\frac{321}{100}$ into $(321\div100=)\ 3\cdot21$, and $\frac{47}{1000}$ into $(47\div1{,}000=)\ \cdot047$] we find $\cdot9+3\cdot21+\cdot047=4\cdot157$.

(*b*.) By means of Fractional Subtraction we find $\frac{53}{100}-\frac{79}{1000}=\frac{530}{1000}-\frac{79}{1000}=\frac{451}{1000}$ or $(451\div1{,}000=)\ \cdot451$; whilst by means of Simple Subtraction [$\frac{53}{100}$ being converted into $(53\div100=)\ \cdot53$, and $\frac{79}{1000}$ into $(79\div1{,}000=)\ \cdot079$] we find $\cdot53-\cdot079=\cdot451$.

(*c*.) By means of Fractional Multiplication we find $\frac{549}{1000}\times\frac{83}{100}=\frac{45567}{100000}$ or $(45{,}567\div100{,}000=)\ \cdot45567$; whilst by means of Simple Multiplication [$\frac{549}{1000}$ being converted into $(549\div 1{,}000=)\ \cdot549$, and $\frac{83}{100}$ into $(83\div100=)\ \cdot83$] we find $\cdot549\times\cdot83=\cdot45567$.

(*d*.) By means of Fractional Division we find $\frac{45567}{100000}\div\frac{83}{100}=\frac{45567}{100000}\times\frac{100}{83}=\frac{4556700}{8300000}=\frac{45567}{83000}$ or $(45567\div83000=45\cdot567\div 83=)\ \cdot549$; whilst by means of Simple Division [$\frac{45567}{100000}$ being converted into $(45567\div100000=)\ \cdot45567$, and $\frac{83}{100}$ into $(83\div 100=)\ \cdot83$] we find $\cdot45567\div\cdot83=(45\cdot567\div83=)\ \cdot549$.

144. To convert a VULGAR fraction into a DECI-

MAL fraction, we *multiply the terms by a number indicating how often the denominator is contained in a power of* 10. The conversion is impossible, however, when the denominator—the vulgar fraction being in its simplest form—contains any *prime* factor which is neither 2 nor 5.

Thus, 5 being a measure of 10, the fraction $\frac{2}{5}$, when its terms are multiplied by (10÷5=) 2, becomes a decimal fraction: $\frac{2}{5}=\frac{2\times2}{5\times2}=\frac{4}{10}$. Again, 4 being a measure—not of 10, but—of 100, the fraction $\frac{3}{4}$, when its terms are multiplied by (100÷4=) 25, becomes a decimal fraction: $\frac{3}{4}=\frac{3\times25}{4\times25}=\frac{75}{100}$. In like manner, 8 being a measure—not of 10, or of 100, but—of 1,000, the fraction $\frac{7}{8}$, when its terms are multiplied by (1,000÷8=) 125, becomes a decimal fraction: $\frac{7}{8}=\frac{7\times125}{8\times125}=\frac{875}{1000}$.

But such fractions as $\frac{5}{7}$, $\frac{8}{15}$, $\frac{15}{22}$, &c., are *not* convertible into decimal fractions—no one of the denominators being a measure of 10, or of any power of 10. In order to understand this, we must remember that (p. 117) "a number which is prime to two or more others is prime to their product;" and that, consequently, *a number which is prime to another is prime to every power of that other.* For instance: 3, being prime to 5, must be prime to each of any number of *fives,* and therefore to their product, 5×5, or $5\times5\times5$, or $5\times5\times5\times5$, &c.; just as 5, being prime to 3, must be prime to each of any number of *threes,* and therefore to their product, 3×3, or $3\times3\times3$, or $3\times3\times3\times3$, &c.

Let us now examine the fraction $\frac{5}{7}$. The denominator, 7, being a prime number, and being neither 2 nor 5 (the prime factors of 10), is prime to both 2 and 5, and to their product, 10, and to every power of 10; so that $\frac{5}{7}$ cannot be converted into a decimal fraction. *Or thus:* If $\frac{5}{7}$ were equal to a decimal fraction with (say) x for numerator, and 10^a for denominator, we should have $\frac{5}{7}=\frac{x}{10^a}$, and $\frac{5\times10^a}{7}=x$. That is, 7 would be a measure of 5×10^a. But this could not be the case: for 7, being prime to 5, and to 10^a (because prime to 10), is prime to 5×10^a.

Neither can $\frac{8}{15}$ be converted into a decimal fraction. Because the denominator, 15 $(=3\times5)$, contains the prime factor 3, which, being neither 2 nor 5, is prime to both 2 and 5, and to $(2\times5=)$ 10, and to every power of 10; and it is obvious that no power of 10 could be a multiple of 15 without being a multiple of 3. *Or thus:* If $\frac{8}{15}$ were equal to a decimal frac-

tion with y for numerator, and 10^b for denominator, we should have $\frac{8}{15}$ or $\frac{8}{3\times 5}=\frac{y}{10^b}$, and $\frac{8\times 10^b}{3\times 5}=y$. That is, 3×5 —and therefore 3—would be a measure of 8×10^b. But this could not be the case: for 3, being prime to 8, and to 10^b (because prime to 10), is prime to 8×10^b.

Nor can $\frac{15}{22}$ be converted into a decimal fraction. Because the denominator, 22 ($=2\times 11$), contains the prime factor 11, which, being neither 2 nor 5, is prime to both 2 and 5, and to ($2\times 5=$) 10, and to every power of 10; and it is impossible for a power of 10 to be measured by 22 without being measured by 11. *Or thus:* If $\frac{15}{22}$ were equal to a decimal fraction with z for numerator, and 10^c for denominator, we should have $\frac{15}{22}$ or $\frac{15}{2\times 11}=\frac{z}{10^c}$, and $\frac{15\times 10^c}{2\times 11}=z$. That is, 2×11—and therefore 11—would be a measure of 15×10^c. But this could not be the case: for 11, being prime to 15, and to 10^c (because prime to 10), is prime to 15×10^c.*

NOTE I.—In determinining whether or not a vulgar fraction is convertible into a decimal fraction, we must take care to see that the fraction is in its *simplest form* before we examine the prime factors of the denominator. The fraction $\frac{12}{15}$, for example, might, at first sight, appear to be not convertible into a decimal fraction, because of the presence, in the denominator, of the prime factor 3; but this factor disappears when the fraction is reduced to its simplest form: $\frac{12}{15}=\frac{12\div 3}{15\div 3}=\frac{4}{5}=\frac{4\times 2}{5\times 2}=\frac{8}{10}$.

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* *General demonstration.*—Let $\frac{x}{y}$ be a vulgar fraction in its simplest form—that is, a vulgar fraction whose terms (x and y) are prime to one another; and let the denominator, y, contain a prime factor, p, which is neither 2 nor 5: to prove that $\frac{x}{y}$ cannot be converted into a decimal fraction. If possible, let $\frac{x}{y}$ be equal to a decimal fraction with w for numerator, and 10^d for denominator; and let p be contained y' times in y. We then have $\frac{x}{y}$ or $\frac{x}{p\times y'}$, $=\frac{w}{10^d}$, and $\frac{x\times 10^d}{p\times y'}=w$. That is, $p\times y'$—and therefore p—is a measure of $x\times 10^d$. But this is impossible, p being prime to x, and to 10^d, and therefore to $x\times 10^d$. That p is prime to x is evident from the fact that x and y—being prime to one another—have no factor in common, and that p is a factor of y. And, that p is prime to 10^d is evident from the fact that p, being a prime number, and being neither 2 nor 5, is prime to both 2 and 5, and to ($2\times 5=$) 10, and to 10^d.

NOTE 2.—A vulgar fraction which—like $\frac{5}{7}$, or $\frac{8}{15}$, or $\frac{15}{22}$—cannot be represented by a decimal fraction of EXACTLY the same value, can, nevertheless, be represented by a decimal fraction of as NEARLY equal value as may be required for any practical purpose: all that is necessary being, to *multiply the terms of the vulgar fraction by a sufficiently high power of* 10, *divide the terms of the resulting fraction by the original denominator, and reject the fractional part of the numerator so obtained.* Multiplying the terms of the fraction $\frac{5}{7}$, for example, by 10, 100, 1,000, 10,000, 100,000, and 1,000,000, successively; dividing the terms of the resulting fraction, in each case, by the original denominator, 7; and rejecting, in each case, the fractional part of the new numerator, we obtain for approximations $\frac{7}{10}$, $\frac{71}{100}$, $\frac{714}{1000}$, $\frac{7142}{10000}$, $\frac{71428}{100000}$, and $\frac{714285}{1000000}$, respectively:—

$$\frac{5}{7}=\frac{5\times 10}{7\times 10}=\frac{50}{70}=\frac{50\div 7}{70\div 7}=\frac{7\frac{1}{7}}{10}$$

$$\frac{5}{7}=\frac{5\times 100}{7\times 100}=\frac{500}{700}=\frac{500\div 7}{700\div 7}=\frac{71\frac{3}{7}}{100}$$

$$\frac{5}{7}=\frac{5\times 1{,}000}{7\times 1{,}000}=\frac{5{,}000}{7{,}000}=\frac{5{,}000\div 7}{7{,}000\div 7}=\frac{714\frac{2}{7}}{1{,}000}$$

$$\frac{5}{7}=\frac{5\times 10{,}000}{7\times 10{,}000}=\frac{50{,}000}{70{,}000}=\frac{50{,}000\div 7}{70{,}000\div 7}=\frac{7{,}142\frac{6}{7}}{10{,}000}$$

$$\frac{5}{7}=\frac{5\times 100{,}000}{7\times 100{,}000}=\frac{500{,}000}{700{,}000}=\frac{500{,}000\div 7}{700{,}000\div 7}=\frac{71{,}428\frac{4}{7}}{100{,}000}$$

$$\frac{5}{7}=\frac{5\times 1{,}000{,}000}{7\times 1{,}000{,}000}=\frac{5{,}000{,}000}{7{,}000{,}000}=\frac{5{,}000{,}000\div 7}{7{,}000{,}000\div 7}=\frac{714{,}285\frac{5}{7}}{1{,}000{,}000}$$

&c. &c.

It will be seen that each of these approximations is closer than the preceding one: the first, $\frac{7}{10}$, being less than the given fraction ($\frac{5}{7}$) by the *tenth* part of $\frac{1}{7}$; the second, $\frac{71}{100}$, by the *hundredth* part of $\frac{3}{7}$; the third, $\frac{714}{1000}$, by the *thousandth* part of $\frac{2}{7}$; the fourth, $\frac{7142}{10000}$, by the *ten-thousandth* part of $\frac{6}{7}$; the fifth, $\frac{71428}{100000}$, by the *hundredth-thousandth* part of $\frac{4}{7}$; and the sixth, $\frac{714285}{1000000}$, by only the *millionth* part of $\frac{5}{7}$. There obviously is no limit to the number of such approximations; and the paradox is thus presented of our being able to go on *continually*

approaching, without ever reaching, such a fraction as $\frac{5}{7}$, which —for a reason already explained—IS NOT EXACTLY equal to any decimal fraction.

145. *Terminate and Interminate Decimals.*— To convert a VULGAR fraction into a simple number of EXACTLY or NEARLY (as the case may be) the same value, we *divide the numerator by the denominator* —reducing units to tenths, tenths to hundredths, hundredths to thousandths, &c. [See NOTE, p. 57.] The decimals thus obtained divide themselves into two classes: (1) those which terminate, and (2) those which do not.

The division of the numerator by the denominator converts $\frac{2}{5}$, $\frac{3}{4}$, and $\frac{7}{8}$, for example, into the "terminate" decimals ·4, ·75, and ·875, respectively:—

5)2·0	4)3·00	8)7·000
·4	·75	·875

If, however, we take $\frac{5}{7}$, or $\frac{8}{15}$, or $\frac{15}{22}$, and divide the numerator by the denominator, we shall—for a reason to be explained presently—obtain an "interminate" decimal; so that, in writing such a fraction as $\frac{5}{7}$ under the form of a simple number, we are obliged to be satisfied with a sufficiently close *approximation.*

146. When the denominator of a fraction—that is, a fraction in its simplest form—contains any prime factor which is neither 2 nor 5, the decimal resulting from the division of the numerator by the denominator will be INTERMINATE; when no such factor occurs, the decimal will be TERMINATE.

This follows from § 144, as we shall find on reflecting that (§ 141) whatever can be converted into a terminate decimal is convertible into a decimal fraction, and that (§ 143) whatever is convertible into a decimal fraction can be converted into a terminate decimal. The fraction $\frac{5}{7}$, or $\frac{8}{15}$, or $\frac{15}{22}$, if convertible into a terminate decimal, could be converted into a decimal fraction; but we have seen that such a fraction as $\frac{5}{7}$, or $\frac{8}{15}$, or $\frac{15}{22}$ is *not* convertible into a decimal fraction. On the other hand, $\frac{2}{5}$, or $\frac{3}{4}$, or $\frac{7}{8}$, being convertible into a decimal fraction, can be converted into a terminate decimal.

147. *Circulating Decimals.*—If carried sufficiently far, every interminate decimal obtained as a quotient would "circulate:" that is, some particular figure, or combination of figures, would be continually repeated.

In order to understand this, let us take the fraction $\frac{5}{7}$, and observe what occurs when the numerator is divided by the denominator. We first reduce the numerator to $(5 \times 10 =)$ 50 tenths, the division of which by 7 gives 7 tenths for quotient, and 1 tenth for remainder: we then reduce the remainder to $(1 \times 10 =)$ 10 hundredths, the division of which by 7 gives 1 hundredth for quotient, and 3 hundredths for remainder: we next reduce this second remainder to $(3 \times 10 =)$ 30 thousandths, the division of which by 7 gives 4 thousandths for quotient, and 2 thousandths for remainder: and so on—the first remainder (1), when a cipher is annexed to it, becoming the second partial dividend (10); the second remainder (3), when a cipher is annexed to it, becoming the third partial dividend (30); the third remainder (2), when a cipher is annexed to it, becoming the fourth partial dividend (20); &c. Now, as the remainder is always less than the divisor, we can never, when dividing by 7, have a larger remainder than 6; and the only other remainders which can possibly arise are 5, 4, 3, 2, and 1. [The remainder can never become 0, because (§ 146) the fraction $\frac{5}{7}$ is not convertible into a terminate decimal.] Annexing a cipher, therefore, to each of these numbers, we find that at every stage of the work—even the very first stage, $\frac{5}{7}$ being a "proper" fraction—the partial dividend must be 60, 50, 40, 30, 20, or 10. So that after setting down the first six figures of the decimal—the list of partial dividends being then exhausted—we are able, without making any calculation, to say with certainty that the seventh partial dividend will be some one of the six already employed, and that, as a necessary result, there will be a repetition of one or more figures in the quotient. The seventh partial dividend being found to be the same as the first, it is evident that—however far we continue the work—the first six partial dividends will be repeated again and again, and in the same order; and that, consequently, the decimal will be a continual repetition of the combination 714285:—

(a)
$\frac{5}{7}$)50(·714285 714285 714285 &c.
49
(b) 10
7
(c) 30
28
(d) 20
14
(e) 60
56
(f) 40
35
(a) 50
49
(b) 10
7
(c) 30
28
(d) 20
14
(e) 60
56
(f) 40
35
(a) 50
49
(b) 10
7
(c) 30
28
(d) 20
14
(e) 60
56
(f) 40
35
&c.

Remainders.		Dividends.
6	(e)	60
5	(a)	50
4	(f)	40
3	(c)	30
2	(d)	20
1	(b)	10

Let us next take the fraction $\frac{8}{11}$, and divide the numerator by the denominator. Before proceeding with the division, we *are able*—after what has already been explained—to say that

not more than (11 − 1 =) 10 quotient figures will have been set down when the decimal begins to circulate. Because, as the remainder will always be 10, 9, 8, 7, 6, 5, 4, 3, 2, or 1, the partial dividend will always be one of these numbers with a cipher annexed: 100, 90, 80, 70, 60, 50, 40, 30, 20, or 10; so that the eleventh partial dividend must be one of the ten previously employed. In the case under consideration, however, we do not exhaust our list of partial dividends; the third being the same as the first, and the first two figures of the quotient, consequently, being continually repeated.

$\frac{8}{11}$)80(·$\widehat{72}$ $\widehat{72}$ $\widehat{72}$ &c.
77
30
22
80
77
30
22
80
77
30
22
&c.

If, as a third illustration, we take the fraction $\frac{2}{3}$, and divide the numerator by the denominator, we shall have only *one* figure (6) continually repeated: $\frac{2}{3}$=·66666 &c.

$\frac{2}{3}$)2·00000 &c.
·66666 &c.

In each of the preceding examples, the decimal circulates from the beginning—the first partial dividend being, in all three cases, the first to repeat itself. Sometimes, however, the decimal circulates (not from the beginning, but) from the second, or third, or fourth, &c. figure—the corresponding partial dividend being the first to repeat itself. The fractions $\frac{15}{22}$, $\frac{7}{12}$, and $\frac{43}{104}$ afford illustrations of this. Converting these fractions into decimals, we find that the circulation begins—in the first case, from the second figure (8); in the second case, from the third figure (3); and in the third case, from the fourth figure (4):—

$\frac{15}{22}$)150(·6 $\widehat{81}$ $\widehat{81}$ $\widehat{81}$ &c.
132
180
176
40
22
180
176
40
22
180
176
40
22
&c.

$\frac{7}{12}$)70(·58333 &c.
60
100
96
40
36
40
36
40
36
&c.

```
43
104)430(·4134615384615 38 &c.
    416
     140                        480
     104                        416
      360                        640
      312                        624
       480                        160
       416                        104
        640                        560
        624                        520
         160                        400
         104                        312
          560                        880
          520                        832
           400                       &c.
           312
            880
            832
```

148. The figure or combination continually repeated in a circulating decimal is called the PERIOD of the decimal.

Thus, in the case of—

·714285 714285 714285 &c.		714285
·72 72 72 &c.		72
·666 &c.	the "period" is	6
·6 81 81 81 &c.		81
·58333 &c.		3
·413 461538 461538 461538 &c.		461538

149. In writing a circulating decimal, we do not, as a rule, proceed farther than the end of the first period, the first and last figures of which we mark by placing a dot over each.*

Thus, for—

$\cdot\overbrace{714285}\,\overbrace{714285}\,\overbrace{714285}$ &c.		$\cdot\dot{7}1428\dot{5}$
$\cdot\overbrace{72}\,\overbrace{72}\,\overbrace{72}$ &c.		$\cdot\dot{7}\dot{2}$
$\cdot 666$ &c.	we write	$\cdot\dot{6}$
$\cdot 6\,\overbrace{81}\,\overbrace{81}\,\overbrace{81}$ &c.		$\cdot 6\dot{8}\dot{1}$
$\cdot 58333$ &c.		$\cdot 58\dot{3}$
$\cdot 413\overbrace{461538}\,\overbrace{461538}\,\overbrace{461538}$ &c.		$\cdot 413\dot{4}6153\dot{8}$

150. Circulating decimals divide themselves into two classes—PURE and MIXED.

151. A circulating decimal is said to be "pure" when the decimal circulates from the beginning: in other words, when the decimal consists exclusively of the period continually repeated.

The following are "pure" circulating decimals: $\cdot\dot{6}$; $\cdot\dot{7}\dot{2}$; $\cdot\dot{7}1428\dot{5}$; &c.

152. A circulating decimal is said to be "mixed" when the decimal does not circulate from the beginning: in other words, when the period (or circulating part) is preceded by a non-circulating part.

The following are "mixed" circulating decimals: $\cdot 6\dot{8}\dot{1}$; $\cdot 58\dot{3}$; $\cdot 413\dot{4}6153\dot{8}$; &c.—the non-circulating parts being $\cdot 6$, $\cdot 58$, and $\cdot 413$, respectively.

* It is hardly necessary to observe that only one dot is employed when the period consists of only one figure.

Conversion of Circulating Decimals into Vulgar Fractions.

Let it be required to convert the pure circulating decimals $\cdot\dot{6}$, $\cdot\dot{7}\dot{2}$, and $\cdot\dot{7}1428\dot{5}$ into vulgar fractions.

Putting a for the fraction equivalent to $\cdot\dot{6}$, we have $a=\cdot666$ &c.; (multiplying by 10) 10 times $a=6\cdot666$ &c.; (taking equals from equals) 10 times $a-a=6\cdot666$ &c. $-\cdot666$ &c.; *i.e.*, 9 times $a=6$; and (dividing by 9) $a=\frac{6}{9}$.

$$\begin{array}{rl} a= & \cdot666 \text{ &c.} \\ 10 \text{ times } a= & 6\cdot666 \text{ &c.} \\ \hline 9 \text{ times } a= & 6 \\ a= & \frac{6}{9} \end{array}$$

II. Putting b for the fraction equivalent to $\cdot\dot{7}\dot{2}$, we have $b=\cdot727272$ &c.; (multiplying by 100) 100 times $b=72\cdot727272$ &c.; (taking equals from equals) 100 times $b-b=72\cdot727272$ &c.$-\cdot727272$ &c.; *i.e.*, 99 times $b=72$; and (dividing by 99) $b=\frac{72}{99}$.

$$\begin{array}{rl} b= & \cdot727272 \text{ &c.} \\ 100 \text{ times } b= & 72\cdot727272 \text{ &c.} \\ \hline 99 \text{ times } b= & 72 \\ b= & \frac{72}{99} \end{array}$$

III. Putting c for the fraction equivalent to $\cdot\dot{7}1428\dot{5}$, we have $c=\cdot714285714285$ &c.; (multiplying by 1,000,000) 1,000,000 times $c=714285\cdot714285714285$ &c.; (taking equals from equals) 1,000,000 times $c-c=714285\cdot714285714285$ &c. $-\cdot714285714285$ &c.; *i.e.*, 999,999 times $c=714285$; and (dividing by 999,999) $c=\frac{714285}{999999}$:—

$$\begin{array}{rl} c= & \cdot714285714285 \text{ &c.} \\ 1{,}000{,}000 \text{ times } c= & 714285\cdot714285714285 \text{ &c.} \\ \hline 999{,}999 \text{ times } c= & 714285 \\ c= & \frac{714285}{999999}. \end{array}$$

We thus find $\cdot\dot{6}=\frac{6}{9}$; $\cdot\dot{7}\dot{2}=\frac{72}{99}$; $\cdot\dot{7}1428\dot{5}=\frac{714285}{999999}$; &c.*

Note.—In the case of a pure circulating decimal, the power of 10 employed as multiplier must contain as many ciphers as the period contains figures—our object being to bring the decimal point to the end of the first period; and it is obvious that the subtraction of 1 from a power of 10 will give as many *nines* for remainder as there are ciphers in the power: $10-1=9$, $100-1=99$, $1{,}000-1=999$, &c.

153. To convert a PURE circulating decimal into a vulgar fraction, we *set down the period—regarded*

* When reduced to their simplest forms, the fractions $\frac{6}{9}$, $\frac{72}{99}$, and $\frac{714285}{999999}$ become $\frac{2}{3}$, $\frac{8}{11}$, and $\frac{5}{7}$, respectively. [See pp. 146–7.]

as a whole number—for numerator; and write for denominator as many NINES *as there are figures in the period.*

Next, let it be required to convert the mixed circulating decimals $\cdot 6\dot{8}\dot{1}$, $\cdot 58\dot{3}$, and $\cdot 413\dot{4}6153\dot{8}$ into vulgar fractions.

I.—Putting x for the fraction equivalent to $\cdot 6\dot{8}\dot{1}$, and multiplying successively by 1,000 and by 10, we have 1,000 times $x = 681\cdot 818181$ &c., and 10 times $x = 6\cdot 818181$ &c.; (taking equals from equals) 1,000 times $x -$ 10 times $x = 681\cdot 818181$ &c. $- 6\cdot 818181$ &c.; *i.e.*, 990 times $x = 681 - 6 = 675$; and (dividing by 990) $x = \frac{675}{990}$.

$$\begin{array}{rrl} & x= & \cdot 681818181\text{ \&c.} \\ 1{,}000\text{ times} & x= & 681\cdot 818181\text{ \&c.} \\ 10\quad ,, & x= & 6\cdot 818181\text{ \&c.} \\ \hline 990\quad ,, & x= & 675 \\ & x= & \frac{675}{990} \end{array}$$

II.—Putting y for the fraction equivalent to $\cdot 58\dot{3}$, and multiplying successively by 1,000 and by 100, we have 1,000 times $y = 583\cdot 333$ &c., and 100 times $y = 58\cdot 333$ &c.; (taking equals from equals) 1,000 times $y -$ 100 times $y = 583\cdot 333$ &c. $- 58\cdot 333$ &c.; *i.e.*, 900 times $y = 583 - 58 = 525$; and (dividing by 900) $y = \frac{525}{900}$.

$$\begin{array}{rrl} & y= & \cdot 58333\text{ \&c.} \\ 1{,}000\text{ times} & y= & 583\cdot 333\text{ \&c.} \\ 100\quad ,, & y= & 58\cdot 333\text{ \&c.} \\ \hline 900\quad ,, & y= & 525 \\ & y= & \frac{525}{900} \end{array}$$

III.—Putting z for the fraction equivalent to $\cdot 413\dot{4}6153\dot{8}$,

$$\begin{array}{rrl} & z= & \cdot 413461538461538\text{ \&c.} \\ 1{,}000{,}000{,}000\text{ times} & z= & 413461538\cdot 461538461538\text{ \&c.} \\ 1{,}000\quad ,, & z= & 413\cdot 461538461538\text{ \&c.} \\ \hline 999{,}999{,}000\quad ,, & z= & 413461125 \\ & z= & \frac{413461125}{999999000} \end{array}$$

and multiplying successively by 1,000,000,000 and by 1,000, we have 1,000,000,000 times $z = 413461538\cdot 461538461538$ &c., and 1,000 times $z = 413\cdot 461538461538$ &c.; (taking equals from equals) 1,000,000,000 times $z -$ 1,000 times $z = 413461538\cdot 461538461538$ &c. $- 413\cdot 461538461538$ &c.; *i.e.*, 999,999,000 times $z = 413461538 - 413 = 413461125$; and (dividing by 999,999,000) $z = \frac{413461125}{999999000}$:—

We thus find $\cdot 6\dot{8}\dot{1} = \dfrac{681-6}{990}$; $\cdot 58\dot{3} = \dfrac{583-58}{900}$; $\cdot 413\dot{4}6153\dot{8} = \dfrac{413461538-413}{999999000}$; &c.*

* When reduced to their simplest forms, the fractions $\frac{675}{990}$, $\frac{525}{900}$, and $\frac{413461125}{999999000}$ become $\frac{15}{22}$, $\frac{7}{12}$, and $\frac{43}{104}$, respectively. [See pp. 147–8.]

NOTE.—In the case of a mixed circulating decimal, we multiply by two different powers of 10, in order to bring the decimal point (*a*) to the end of the first period, and (*b*) to the end of the non-circulating part of the decimal, respectively. So that the lower power of 10 contains as many ciphers as there are figures in the non-circulating part of the decimal, whilst the higher power contains as many *additional* ciphers as there are figures in the period. The following § is thus rendered intelligible when we reflect that, in taking a lower from a higher power of 10, we begin by setting down as many ciphers as there are in the lower power, and we finish by writing as many *nines* as there are additional ciphers in the higher power.

154. To convert a MIXED circulating decimal into a vulgar fraction: From the decimal, in its contracted form, (*i.e.*, to the end of the first period,) subtract the non-circulating part—regarding both as whole numbers, and set down the remainder for numerator; then write, for denominator, as many *nines* as there are figures in the period, and—after those nines—as many ciphers as there are figures in the non-circulating part.

NOTE.—When the denominator of a vulgar fraction which is in its simplest form contains amongst its prime factors either 2 or 5, as well as one or more other prime factors different from 2 and 5, the division of the numerator by the denominator will give a MIXED circulating decimal; and the number of figures in the non-circulating part will be equal to the number of *twos** or of *fives**—as the case may be—found amongst the prime factors of the denominator. So that the vulgar fractions from which PURE circulating decimals are obtained are those in whose denominators—the fractions being in their simplest forms—neither 2 nor 5 occurs as a factor.

This is rendered intelligible by the following facts:—

(I.) A number which measures the product of two factors, and is prime to one of them, measures the other factor.

(II.) The terms of a fraction which is in its simplest form are measures, respectively, of the terms of every other fraction of equal value.

These facts are easily explained.

(I.) Let m measure $x \times y$, and be prime to x: to prove that

* The number of TWOS, when the *fives* are not more numerous; the number of FIVES, when the *twos* are not more numerous; and of course *the number of* EACH (not of *both*) when 2 and 5 occur equally often.

m measures y. As m measures $x \times y$, the prime factors of m all occur amongst those of $x \times y$—that is, amongst the prime factors of x and those of y, taken collectively. But, m being prime to x, the prime factors of m do not, any of them, occur amongst those of x; therefore the prime factors of m must all occur amongst those of y—in other words, m must measure y.

(II.) Let $\frac{a}{b}=\frac{A}{B}$, and let $\frac{a}{b}$ be in its simplest form: to prove that a measures A, and that b measures B. Multiplying by B, we have $\frac{a \times B}{b}=A$; so that b measures $a \times B$. But b is prime to a, the fraction $\frac{a}{b}$ being in its simplest form; consequently (I.) b measures B. Next, by inverting the given fractions, or dividing unity by each, we have $\frac{b}{a}=\frac{B}{A}$, and (multiplying by A) $\frac{b \times A}{a}=B$; so that a measures $b \times A$. But a is prime to b; therefore (I.) a measures A.

Let us now take $\frac{19}{21}$ as an illustration of a fraction in its simplest form, and in whose denominator neither 2 nor 5 occurs as a factor. For a reason already explained, this fraction will become a circulating decimal—either "pure" or "mixed" —when the numerator is divided by the denominator. If convertible into a pure circulating decimal, $\frac{19}{21}$ is (§ 153) equivalent to a fraction having some number of *nines* for denominator; whilst (§ 154), if convertible into a mixed circulating decimal, $\frac{19}{21}$ is equivalent to a fraction having for denominator some number of *nines* followed by some number of ciphers—in other words, the product of some number of *nines* by a power of 10. In the latter case, 21 would (II.) measure the product of some number of *nines* by a power of 10—the fraction $\frac{19}{21}$ being in its simplest form. But (I.) 21 could not measure such a product without measuring the *nines:* because, as neither 2 nor 5 occurs amongst its prime factors, 21 is prime to 10, and to every power of 10. So that $\frac{19}{21}$ is equivalent to a fraction having some number of *nines* for denominator; and as (§ 153) every such fraction is the equivalent of a pure circulating decimal having the figure or figures of the numerator for period, it follows that $\frac{19}{21}$ is convertible into a *pure*—not a mixed—circulating decimal.

$$\left\{\begin{array}{l}
\frac{13}{11\times2}=\frac{13}{11}\times\frac{1}{2}=\frac{13}{11}\times\frac{5}{2\times5}=\frac{13\times5}{11}\times\frac{1}{10}=\frac{65}{11}\div10=5\tfrac{10}{11}\div10=5\cdot\dot{9}\dot{0}\div10=\cdot5\dot{9}\dot{0}\\[2ex]
\frac{13}{11\times5}=\frac{13}{11}\times\frac{1}{5}=\frac{13}{11}\times\frac{2}{5\times2}=\frac{13\times2}{11}\times\frac{1}{10}=\frac{26}{11}\div10=2\tfrac{4}{11}\div10=2\cdot\dot{3}\dot{6}\div10=\cdot2\dot{3}\dot{6}\\[2ex]
\frac{13}{11\times2\times5}=\frac{13}{11}\times\frac{1}{2\times5}=\frac{13}{11}\div10=1\tfrac{2}{11}\div10=1\cdot\dot{1}\dot{8}\div10=\cdot1\dot{1}\dot{8}
\end{array}\right.$$

$$\left\{\begin{array}{l}
\frac{7}{3\times2\times2}=\frac{7}{3}\times\frac{1}{2\times2}=\frac{7}{3}\times\frac{5\times5}{2\times2\times5\times5}=\frac{7\times5\times5}{3}\times\frac{1}{100}=\frac{175}{3}\div100=58\tfrac{1}{3}\div100=58\cdot\dot{3}\div100=\cdot58\dot{3}.\\[2ex]
\frac{7}{3\times5\times5}=\frac{7}{3}\times\frac{1}{5\times5}=\frac{7}{3}\times\frac{2\times2}{5\times5\times2\times2}=\frac{7\times2\times2}{3}\times\frac{1}{100}=\frac{28}{3}\div100=9\tfrac{1}{3}\div100=9\cdot\dot{3}\div100=\cdot09\dot{3}.\\[2ex]
\frac{7}{3\times2\times2\times5}=\frac{7}{3}\times\frac{1}{2\times2\times5}=\frac{7}{3}\times\frac{5}{2\times2\times5\times5}=\frac{7\times5}{3}\times\frac{1}{100}=\frac{35}{3}\div100=11\tfrac{2}{3}\div100=11\cdot\dot{6}\div100=\cdot11\dot{6}.\\[2ex]
\frac{7}{3\times2\times5\times5}=\frac{7}{3}\times\frac{1}{2\times5\times5}=\frac{7}{3}\times\frac{2}{2\times2\times5\times5}=\frac{7\times2}{3}\times\frac{1}{100}=\frac{14}{3}\div100=4\tfrac{2}{3}\div100=4\cdot\dot{6}\div100=\cdot04\dot{6}.\\[2ex]
\frac{7}{3\times2\times2\times5\times5}=\frac{7}{3}\times\frac{1}{2\times2\times5\times5}=\frac{7}{3}\times\frac{1}{100}=2\tfrac{1}{3}\div100=2\cdot\dot{3}\div100=\cdot02\dot{3}.
\end{array}\right.$$

$$\frac{3}{7\times2\times2\times2}=\frac{3}{7}\times\frac{1}{2\times2\times2}=\frac{3}{7}\times\frac{5\times5\times5}{2\times2\times2\times5\times5\times5}=\frac{3\times5\times5\times5}{7}\times\frac{1}{1000}=\frac{375}{7}\div1000=53\tfrac{4}{7}\div1000=53\cdot\dot{5}7142\dot{8}\div1000=\cdot053\dot{5}7142\dot{8}.$$

$$\frac{3}{7\times5\times5\times5}=\frac{3}{7}\times\frac{1}{5\times5\times5}=\frac{3}{7}\times\frac{2\times2\times2}{5\times5\times5\times2\times2\times2}=\frac{3\times2\times2\times2}{7}\times\frac{1}{1000}=\frac{24}{7}\div1000=3\tfrac{3}{7}\div1000=3\cdot\dot{4}2857\dot{1}\div1000=\cdot003\dot{4}2857\dot{1}.$$

$$\frac{3}{7\times2\times2\times2\times5}=\frac{3}{7}\times\frac{1}{2\times2\times2\times5}=\frac{3}{7}\times\frac{5\times5}{2\times2\times2\times5\times5\times5}=\frac{3\times5\times5}{7}\times\frac{1}{1000}=\frac{75}{7}\div1000=10\tfrac{5}{7}\div1000=10\cdot\dot{7}1428\dot{5}\div1000=\cdot010\dot{7}1428\dot{5}.$$

$$\frac{3}{7\times2\times5\times5\times5}=\frac{3}{7}\times\frac{1}{2\times5\times5\times5}=\frac{3}{7}\times\frac{2\times2}{2\times2\times2\times5\times5\times5}=\frac{3\times2\times2}{7}\times\frac{1}{1000}=\frac{12}{7}\div1000=1\tfrac{5}{7}\div1000=1\cdot\dot{7}1428\dot{5}\div1000=\cdot001\dot{7}1428\dot{5}.$$

$$\frac{3}{7\times2\times2\times2\times5\times5}=\frac{3}{7}\times\frac{1}{2\times2\times2\times5\times5}=\frac{3}{7}\times\frac{5}{2\times2\times2\times5\times5\times5}=\frac{3\times5}{7}\times\frac{1}{1000}=\frac{15}{7}\div1000=2\tfrac{1}{7}\div1000=2\cdot\dot{1}4285\dot{7}\div1000=\cdot002\dot{1}4285\dot{7}.$$

$$\frac{3}{7\times2\times2\times5\times5\times5}=\frac{3}{7}\times\frac{1}{2\times2\times5\times5\times5}=\frac{3}{7}\times\frac{2}{2\times2\times2\times5\times5\times5}=\frac{3\times2}{7}\times\frac{1}{1000}=\tfrac{6}{7}\div1000=\cdot\dot{8}5714\dot{2}\div1000=\cdot000\dot{8}5714\dot{2}.$$

$$\frac{3}{7\times2\times2\times2\times5\times5\times5}=\frac{3}{7}\times\frac{1}{2\times2\times2\times5\times5\times5}=\frac{3}{7}\times\frac{1}{1000}=\frac{3}{7}\div1000=\cdot\dot{4}2857\dot{1}\div1000=\cdot000\dot{4}2857\dot{1}.$$

Let us next take the fraction $\frac{15}{22}$, which is in its simplest form, and the prime factors of whose denominator are 2 and 11. This fraction will become either a pure or a mixed circulating decimal, when the numerator is divided by the denominator. Now, $\frac{15}{22}$, if convertible into a pure circulating decimal, would be equivalent to a fraction having some number of *nines* for denominator; and this denominator would be measured by 22, and therefore by 2, a factor of 22. But 2 is not a measure of any such denominator (9, or 99, or 999, &c.); and consequently the circulating decimal must be *mixed*—not pure.

Again: $\frac{14}{15}$, the prime factors of whose denominator are 3 and 5, will, when the numerator is divided by the denominator, become either a pure or a mixed circulating decimal. Now, $\frac{14}{15}$, if convertible into a pure circulating decimal, would be equivalent to a fraction having some number of *nines* for denominator, and this denominator would be measured by 15, and therefore by 5, a factor of 15. But 5 is not a measure of 9, or of 99, or of 999, &c.—the division of every such number by 5 leaving 4 for remainder; consequently the circulating decimal must, as in the last case, be *mixed*—not pure.

From an examination of the examples in the two preceding pages, it will be seen that, in the case of a fraction convertible into a mixed circulating decimal, the number of figures in the non-circulating part of the decimal is always indicated by the number of *twos* or of *fives* (see p. 152) found amongst the prime factors of the denominator, when the fraction is in its simplest form. In each case we first obtain (every such fraction as $\frac{13\times5}{11}$, $\frac{13\times2}{11}$, &c. producing) a PURE circulating decimal, and this we afterwards convert into a MIXED circulating decimal by removing the decimal point to the left: the number of places the point is removed—and therefore the number of places occupied by the non-circulating part of the decimal—being the same as the number of *twos* or of *fives* found amongst the prime factors of the given fraction's denominator. Because the lowest power of 10 which is measured by the product of (I.) a number of *twos* (II.), a number of *fives*, or (III.) a number of *twos* AND *fives*, must contain as many ciphers as there are *twos* or *fives*: $10=2\times5$; $100=2\times2\times5\times5$; $1{,}000=2\times2\times2\times5\times5\times5$; &c

PROPORTION.

Simple Proportion, or the "Rule of Three."

155. When we speak of a number as "large" or "small," we necessarily compare it—although unconsciously, sometimes—to another number of the *same kind.*

A shilling, for example, is *large* in amount compared to a penny, but *small* compared to a sovereign; a rood of land is *large* compared to a perch, but *small* compared to an acre; a quart is *large* compared to a pint, but *small* compared to a gallon; and so on. In like manner, 10 pounds of tea are *large* in quantity compared to 2 pounds, but *small* compared to 80 pounds; a piece of cloth 12 yards long is *large* compared to a piece of the same cloth 4 yards long, but *small* compared to a piece 30 yards long; a flock of 20 birds is *large* compared to a flock of 6, but *small* compared to a flock of 100; &c.

156. The relation—as to largeness or smallness—of one number to another of the same kind, is called the RATIO of the one number to the other.

Thus, the relation—as to largeness—of 10 pounds of tea to 2 pounds, is called the "ratio" of 10 pounds to 2 pounds; the relation—as to smallness—of 12 yards of cloth to 30 yards, the "ratio" of 12 yards to 30 yards; and so on.

157. Of two numbers so compared—and which are spoken of as the *terms* of the ratio—the first is called the ANTECEDENT; the second, the CONSEQUENT.

In the case of the ratio of 10 pounds to 2 pounds, the "antecedent" is 10 pounds, and the "consequent" 2 pounds; in the case of the ratio of 12 yards to 30 yards, the "antecedent" is 12 yards, and the "consequent" 30 yards; &c.

158. The terms of a ratio are written one after the other,—the consequent after the antecedent,—and separated by two dots, placed one below the other.

Thus, the ratio of

$$\left.\begin{array}{l}\text{10 lbs. to 2 lbs.}\\ \text{12 yds. ,, 30 yds.}\end{array}\right\}\text{ would be written }\left\{\begin{array}{l}\text{10 lbs. : 2 lbs.}\\ \text{12 yds. : 30 yds.}\end{array}\right.$$

&c. &c.

159. By the VALUE of a ratio is meant—the number indicating *how* large or small the antecedent is, compared to the consequent.

How large (in amount) is 1*s.*, compared to 4*d.*? This question can be answered in either of two ways—according as we employ Division or Subtraction: (*a*) 1*s.* is *three times* larger than 4*d.*; (*b*) 1*s.* exceeds 4*d.* by 8*d.* How small is 1*s.*, compared to £1? This question, like the last, admits of two answers: (*a*) 1*s.* is *twenty times* smaller than, or is the *twentieth* part of, £1; (*b*) 1*s.* is less than £1 by 19*s.*

In the case of a RATIO, however, we invariably employ Division—not Subtraction. The value of the ratio 1*s.* : 4*d.* is not 8*d.*, but (1*s.* ÷ 4*d.* =) 3; and the value of the ratio 1*s.* : £1 is not 19*s.*, but (1*s.* ÷ £1 =) $\frac{1}{20}$.

160. The "value" of a ratio is the quotient resulting from the division of the antecedent by the consequent.

Thus, the value of the ratio—

$$\left.\begin{array}{r} 10 : 2 \\ 2 : 10 \\ 15 : 5 \\ 5 : 15 \\ 30 : 12 \\ 12 : 30 \end{array}\right\} \text{ is } \left\{\begin{array}{l} \frac{10}{2} = 5 \\ \frac{2}{10} = \frac{1}{5} \\ \frac{15}{5} = 3 \\ \frac{5}{15} = \frac{1}{3} \\ \frac{30}{12} = 2\frac{1}{2} \\ \frac{12}{30} = \frac{2}{5} \end{array}\right.$$

&c. &c.

To say, therefore, that the value of a ratio is 5, or 3, or $2\frac{1}{2}$, is merely to convey that the antecedent is 5 times, or 3 times, or $2\frac{1}{2}$ times the consequent—as the case may be; just as, in saying that the value of a ratio is $\frac{1}{5}$, or $\frac{1}{3}$, or $\frac{2}{5}$, we mean that the antecedent is $\frac{1}{5}$, or $\frac{1}{3}$, or $\frac{2}{5}$—as the case may be— of the consequent. It is hardly necessary to add, that the value of a ratio is always an *abstract* number.

Every two ratios which we compare—the one to the other—are either equal or unequal in value. Thus, 10 : 2 and 15 : 5 are unequal ratios, the value of 10 : 2 being ($\frac{10}{2}$=)5, whilst that of 15 : 5 is ($\frac{15}{5}$=)3. So that 10 compared to 2 is a larger number than 15 compared to 5; or, as a school-boy would express it, the *number of times* 10 is larger than 2 is greater than the *number of times* 15 is larger than 5. On the other hand, the ratios 10 : 2 and 15 : 3—the value of each being 5—are equal: $\frac{10}{2}=5$; $\frac{15}{3}=5$. The ratios 4 : 12 and 7 : 21, also,—the value of each being $\frac{1}{3}$,—are equal: $\frac{4}{12}=\frac{1}{3}$; $\frac{7}{21}=\frac{1}{3}$.

161. When two equal ratios are written one after the other, with the sign of equality (=) between them,—this sign, however, being usually represented by four dots (: :),—the expression is called a PROPORTION.

So that a PROPORTION is *an expression indicating that two ratios are equal.* Each of the following expressions is a "Proportion:"

$$\left.\begin{array}{l}\text{(I.) } 10:\ 2=15:\ 3\\ \text{(II.) } 4:12=\ 7:21\end{array}\right\}\text{ more usually written }\left\{\begin{array}{l}\text{(I.) } 10:\ 2::15:\ 3\\ \text{(II.) } 4:12::\ 7:21\end{array}\right.$$

The preceding Proportions would be read in this way—the words in brackets, however, being omitted:

(I.) As [large as] 10 is [compared] to 2, so [large] is 15 [compared] to 3.

(II.) As [small as] 4 is [compared] to 12, so [small] is 7 [compared] to 21.

162. Of the four numbers—or "terms"—which occur in a Proportion, the first and last are called the EXTREMES; the second and third, the MEANS.

Thus, in the Proportion 10 : 2 :: 15 : 3 the "extremes" are 10 and 3; the "means," 2 and 15. And in the Proportion 4 : 12 :: 7 : 21 the extremes are 4 and 21; the means, 12 and 7.

163. *The product of the extremes is, in every case, equal to the product of the means.*

Taking the Proportion 10 : 2 :: 15 : 3, we see that the value of the first ratio (10 : 2) is $\frac{10}{2}$; that the value of the second (15:3) is $\frac{15}{3}$; and that, the two ratios being equal, the two fractions must be equal. Multiplying both fractions by 2, and the resulting products by 3, (2 and 3 being the denominators,) we have $10=\frac{2\times15}{3}$, and $10\times3=2\times15$: *i.e.*, the product of the extremes = the product of the means.

$$10:2::15:3$$
$$\frac{10}{2}=\frac{15}{3}$$
$$10=\frac{2\times15}{3}$$
$$10\times3=2\times15$$

Next, taking the Proportion 4 : 12 :: 7 : 21, we see that the value of the first ratio (4 : 12) is $\frac{4}{12}$; that the value of the second (7 : 21) is $\frac{7}{21}$; and that, as the two ratios are equal, the two fractions must be equal. Multiplying both fractions by 12, and the resulting products by 21, (12 and 21 being the two denominators,) we have $4=\frac{12\times7}{21}$, and $4\times21=12\times7$: *i.e.*, as before, the product of the extremes = the product of the means.

$$4:12::7:21$$
$$\frac{4}{12}=\frac{7}{21}$$
$$4=\frac{12\times7}{21}$$
$$4\times21=12\times7$$

General demonstration.—Let the Proportion be $w : x :: y : z$.

The value of the first ratio ($w : x$) is $\frac{w}{x}$; the value of the second ($y : z$), $\frac{y}{z}$. The two ratios being equal, the two fractions are equal. Multiplying both fractions by x, and the resulting products by z, (x and z being the two denominators,)

$$w : x :: y : z$$
$$\frac{w}{x} = \frac{y}{z}$$
$$w = \frac{x \times y}{z}$$
$$w \times z = x \times y$$

we have $w = \frac{x \times y}{z}$, and $w \times z = x \times y$: *i.e.*, as in each of the preceeding cases, the product of the extremes=the product of the means.

164. A knowledge of the fact that "the product of the extremes is equal to the product of the means" enables us to find any term of a Proportion, when the remaining three terms are given. On this account, Proportion is—and always has been—commonly known as the "Rule of Three."

Let it be required to find the absent term in each of the following cases:

(I.) ? : 4 :: 18 : 6
(II.) 2 : ? :: 5 : 20
(III.) 11 : 7 :: ? : 14
(IV.) 8 : 10 :: 2 : ?

(I.) The product of the means (4×18) being 72, the product of the extremes also is 72; and as one of the extremes is 6, the other must be $72 \div 6 = 12$.

(II.) The product of the extremes (2×20) being 40, the product of the means also is 40; and as one of the means is 5, the other must be $40 \div 5 = 8$.

(III.) The product of the extremes (11×14) being 154, the product of the means also is 154; and as one of the means is 7, the other must be $154 \div 7 = 22$.

(IV.) The product of the means (10×2) being 20, the product of the extremes also is 20; and as one of the extremes is 8, the other must be $20 \div 8 = 2\frac{1}{2}$.

The absent terms being represented by *a*, *b*, *c*, and *d*, respectively, we thus have—

(I.) $a : 4 :: 18 : 6$; $a = 4 \times 18 \div 6$
(II.) $2 : b :: 5 : 20$; $b = 2 \times 20 \div 5$
(III.) $11 : 7 :: c : 14$; $c = 11 \times 14 \div 7$
(IV.) $8 : 10 :: 2 : d$; $d = 10 \times 2 \div 8$

165. To find any term of a Proportion, when the remaining three are known: If the required term be one of the extremes [see above, I. and IV.], divide the product of the means by the given extreme; if the term required be one of the means [see II. and III.], divide the product of the extremes by the given mean.

Let us now apply our knowledge of Proportion to a few practical examples.

EXAMPLE I.—The price of 4 pounds of tea being 18 shillings, what is the price of 10 pounds?

It is obvious that, 10 pounds being a larger quantity than 4 pounds, the price of 10 pounds is a larger amount than the price of 4 pounds. It is equally obvious that, if 10 pounds were the double of 4 pounds, the price of 10 pounds would be twice as much as the price of 4 pounds; that, if 10 pounds were the treble of 4 pounds, the price of 10 pounds would be three times as much as the price of 4 pounds; and so on. As large in quantity as 10 pounds are, compared to 4 pounds, so large in amount is the price of 10 pounds, compared to the price of 4 pounds: in other words, the *ratio* of 10 pounds to 4 pounds is equal to the *ratio* of the price of 10 pounds to the price of 4 pounds.*

Here, then, is a Proportion, of which three terms are given, and one is required. The question being "noted" in the way shown in the margin, and *a* being put for the "answer," we can arrange the terms of the proportion in any one of four ways:—

lbs.	s.
4—	—18
10—	— ?

(I.) Making *a* (which represents *money*) the first term, we necessarily make 18*s*. the second term: because the first two terms of a Proportion—being the terms of the first ratio—must (§ 156) be of the *same kind*. Then, as the antecedent of this ratio is larger than the consequent,—the price of 10 lbs. being evidently more than the price of 4 lbs.,—the antecedent of the

* Of course it would be equally correct to say—as *small* in quantity as 4 pounds are, compared to 10 pounds, so *small* in amount is the price of 4 pounds, compared to the price of 10 pounds: in other words—the "ratio" of 4 pounds to 10 pounds is equal to the "ratio" of the price of 4 pounds to the price of 10 pounds.

second ratio also must be larger than the consequent. We therefore write 10 lbs. in the third, and 4 lbs. in the fourth place:

$$\begin{array}{cccc} & s. & \text{lbs.} & \text{lbs.} \\ a : & 18 :: & 10 : & 4 \end{array}$$

(II.) Making a the second term, we necessarily make 18*s.* the first term. Then, as the antecedent of this ratio is smaller than the consequent, the antecedent of the second ratio also must be smaller than the consequent. We therefore write 4 lbs. in the third, and 10 lbs. in the fourth place:

$$\begin{array}{cccc} s. & & \text{lbs.} & \text{lbs.} \\ 18 : & a :: & 4 : & 10 \end{array}$$

(III.) Making a the third term, we necessarily make 18*s.* the fourth term: because the last two terms of a Proportion—being the terms of the second ratio—must be of the *same kind.* Then, the antecedent of this ratio being the larger term of the two, the antecedent of the first ratio also must be the larger term of the two. We therefore write 10 lbs. in the first, and 4 lbs. in the second place:

$$\begin{array}{cccc} \text{lbs.} & \text{lbs.} & & s. \\ 10 : & 4 :: & a : & 18 \end{array}$$

(IV.) Lastly, making a the fourth term, we necessarily make 18*s.* the third term. Then, the antecedent of this ratio being the smaller term of the two, the antecedent of the first ratio also must be the smaller term of the two. We therefore write 4 lbs. in the first, and 10 lbs. in the second place:

$$\begin{array}{cccc} \text{lbs.} & \text{lbs.} & s. & \\ 4 : & 10 :: & 18 : & a \end{array}$$

From any one of these Proportions we find $a = (18 \times 10 \div 4 =)$ 45*s.* In practice, however, the "answer" is always made the *fourth* term.

Example II.—The yearly rent of 3 acres of land being £5 13*s.* 6*d.*, how much a year ought to be paid for 29A. 1R. 17P. of the same land?

Here we have to find a sum of money as many times larger than £5 13*s.* 6*d.* as 29A. 1R. 17P. is larger than 3A.: in other words, a sum that bears to £5 13*s.* 6*d.* the ratio which 29A. 1R. 17P. bears to 3A.; or, a sum to which £5 13*s.* 6*d.* bears the ratio that 3A. bears to 29A 1R. 17P.

A.	R.	P.		£	s.	d.
3			———	5	13	6
29	1	17	———	?		

Putting *a* for the required amount, and making it the fourth term of the Proportion, we necessarily—for the reason already explained—make £5 13*s.* 6*d.* the third term. Then, as the antecedent of this ratio is the smaller term of the two (the antecedent being the rent of 3A., whilst the consequent is the rent of 29A. 1R. 17P.), the antecedent of the first ratio also must be the smaller term of the two. We therefore write 3A. in the first, and 29A. 1R. 17P. in the second place:

```
A.   A.  R. P.    £  s.  d.
3 : 29   1 17 : : 5  13  6 : a
```

When the first two terms are reduced to the same denomination (perches), and the third term to pence, the Proportion becomes

```
 P.     P.       d.
480 : 4697 : : 1362 : a ;
```

and from this Proportion we find $a = (4697 \times 1362 \div 480 =)$ 13328*d.*, or £55 10*s.* 8*d.*, nearly.

Here is the work in full:

```
 A.        A.  R. P.       £  s.  d.
 3    :    29  1  17  : :  5  13  6  :  a
 4          4             20
 12       117            113
 40        40             12
480      4697           1362
         1362
         9394
        28182
       14091
       4697
  480)6397314(13328
       480
       1597
       1440                  12)13328d.
        1573                  20)1110s. 8d.
        1440                 a=£55 10s. 8d.
         1331
          960
          3714
          3840
```

EXAMPLE III.—If 12 men, working 10 hours a day, can reap a field of corn in 14 days, in what time could 8 men, working 10 hours a day, reap it?

An inexperienced pupil would probably be puzzled by the occurrence, in this question, of four different numbers. A little reflection, however, makes it evident that—being the same in both cases—the length of the day has nothing to say to the answer, and may therefore be dismissed from consideration. The "10 hours a day" being omitted, the question assumes this less embarrassing form:

	men.	days.
If 12 men can reap a field of corn in 14 days, in how many days (of the *same length*) could 8 men reap it?	12	14
	8	?

The LARGER the number of men employed to do a certain quantity of work, the SMALLER the number of days required for the performance of the work; and *vice versa*. What 12 men, for instance, can do in a certain time, 24 men could do in *half* the time; 36 men, in a *third* of the time; 48 men, in a *fourth* of the time; &c. On the other hand, what 12 men can do in a certain time, 6 men could do in *double* the time; 4 men, in *treble* the time; 3 men, in *quadruple* the time; &c.

Returning to the last example, we see that, 8 men being SMALLER in number than 12 men, the days required by 8 men to reap the field of corn must be LARGER in number than the days required by 12 men; and that, as SMALL in number as 8 men are, compared to 12 men, so LARGE in number are the days required by 8 men, compared to the days required by 12 men. In other words, as SMALL in number as 8 men are, compared to 12 men, so SMALL in number are the days required by 12 men, compared to the days required by 8 men: *i.e.*, the ratio of 8 men to 12 men is equal to the ratio of the time required by 12 men to the time required by 8 men.

In setting down this Proportion, we proceed as in the other cases. Putting a for the answer, and making it the fourth term, we make 14 days the third term. Then, as the antecedent of this ratio is the smaller term of the two (the antecedent being the time required by 12 men, whilst the consequent is the time required by 8 men), the antecedent of the first ratio also must be the smaller term of the two. We therefore write 8 men in the first, and 12 men in the second place:

$$\begin{array}{ccc} \text{men.} & \text{men.} & \text{days.} \\ 8 : & 12 :: & 14 : a; \end{array}$$

and from this Proportion we find $a=(14\times 12\div 8=)$ 21 days.

EXAMPLE IV.—A piece of cloth being 30 yards long and $\frac{7}{12}$ of a yard broad, what is the length of another piece $\frac{5}{8}$ of a yard in breadth, and containing the same quantity of cloth?

As the quantity of cloth is the same in both cases, the broader piece must be the shorter of the two, and must be *as many times* shorter as it is wider than the other piece. Reducing $\frac{7}{12}$ and $\frac{5}{8}$ to a common denominator, we find $\frac{5}{8}$ to be the larger fraction:

yds. lg.		yds. bd.
30	———	$\frac{7}{12}$
?	———	$\frac{5}{8}$

$\frac{7}{12}=\frac{7\times 2}{12\times 2}=\frac{14}{24}$; $\frac{5}{8}=\frac{5\times 3}{8\times 3}=\frac{15}{24}$. So that the length of the piece $\frac{5}{8}$ of a yard broad must be less than 30 yards.

Putting a for the answer, and making it the fourth term, we make 30 yards the third term. Then, the antecedent of this ratio being greater than the consequent, the antecedent of the first ratio also must be greater than the consequent; for which reason we write $\frac{5}{8}$ in the first, and $\frac{7}{12}$ in the second place:

yd.		yd.		yds.		
$\frac{5}{8}$	:	$\frac{7}{12}$	::	30	:	a

From this Proportion we find $a=(30\times\frac{7}{12}\div\frac{5}{8}=30\times\frac{7}{12}\times\frac{8}{5}=\frac{1680}{60}=)28$ yards.*

166. Rule for the working of exercises in (Simple) Proportion: (I.) Leaving the fourth place for the answer, set down as the third term the number which is of the same kind as the answer. (II.) Make the remaining two numbers the terms of the first ratio—writing the larger number in the first, and the smaller in the second place, if the third term be greater than the answer;† but writing the smaller number in the first, and the larger in the second place, if the third term be less than the answer.† (III.) Should the first two terms not be simple numbers of the same denomination, reduce

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* Having reduced $\frac{5}{8}$ and $\frac{7}{12}$ to the forms $\frac{15}{24}$ and $\frac{14}{24}$, respectively, we save ourselves unnecessary trouble by making 15 yards the first, and 14 yards the second term of the Proportion. Because it is evident, from what has already been explained, that a ratio—like a fraction—remains unaltered in value when the terms are both multiplied, or both divided, by the same number; and the ratio $\frac{15}{24}:\frac{14}{24}$, when its terms are multiplied by 24, becomes 15 : 14.

yd. yd. yds.
$\frac{15}{24}:\frac{14}{24}::30:a$
$15:14::30:a$
$a=(30\times 14\div 15=$ 28 yards.

† The nature of the question enables us at once to say, in every case whether the third term, compared to the answer, is *large* or *small*.

them to the same denomination; and should the third term be a compound number, reduce it to a simple number.* (IV.) Then, treating all three as abstract numbers, divide the product of the second and third terms by the first term; and the quotient will be the answer—its denomination being understood to be the same as that to which the third term has been reduced.

NOTE.—Every Proportion whose terms are concrete numbers is either a DIRECT or an INVERSE† Proportion. The distinction will be rendered intelligible by a reference to the preceding examples, the first two of which are "direct;" the last two, "inverse." Before reverting to the examples, however, we must understand (*a*) that two numbers are said to vary DIRECTLY when they are so related to each other that any increase of the one produces a proportionate increase of the other, and *vice versa;* also (*b*), that two numbers are said to vary INVERSELY when they are so related that any *increase* of the one produces a proportionate *decrease* of the other; and *vice versa.* For instance:

(*a.*) In an exercise in Multiplication, the multiplicand and the product vary DIRECTLY—the multiplier remaining unaltered: because the larger the multiplicand, the larger the product; and the smaller the multiplicand, the smaller the product.

(*b.*) In an exercise in Division, the divisor and the quotient vary INVERSELY—the dividend remaining unaltered: because the *larger* the divisor the *smaller* the quotient; and the *smaller* the divisor, the *larger* the quotient.

Let us now return to the examples:

(I.) Tea and its price—all other circumstances being the same—vary DIRECTLY: the larger the quantity, the larger the price; and the smaller the quantity, the smaller the price. So that when we set down the ratio of 4 lbs. to 10 lbs., and the ratio of the price of 4 lbs. to the price of 10 lbs., we have two equal ratios, which—when the equality is expressed—constitute a DIRECT Proportion.

$$\begin{array}{l} \text{lbs. lbs.} \quad\quad s. \\ 4 : 10 :: 18 : a \end{array}$$

---

* The reduction of the third term is sometimes unnecessary. Here is an illustration:

$$\begin{array}{rrrrrr} \text{yds.} & \text{yds.} & £ & s. & d. & \\ 7 : & 11 :: & 2 & 14 & 3 & : a \\ & & & & 11 & \\ \hline & 7) & 29 & 16 & 9 & \\ \hline & a = & 4 & 5 & 3 & \end{array}$$

† *Instead of* "inverse," we sometimes say "INDIRECT."

(II.) Land and the rent paid for it—all other circumstances being the same—vary DIRECTLY: the larger the quantity, the higher the rent; and the smaller the quantity, the lower the rent. So that when we set down the ratio of 3 acres to 29A. 1R. 17P., and the ratio of the rent of 3 acres to the rent of 29A. 1R. 17P., we

A.		A.	R.	P.		£	s.	d.		
3	:	29	1	17	::	5	13	6	:	*a*

have two equal ratios, which—when the equality is expressed—constitute a DIRECT Proportion.

(III.) The number of men employed, and the number of days occupied, in the performance of a certain quantity of work —all other circumstances being the same—vary INVERSELY: the *larger* the number of men, the *smaller* the number of days; and the *smaller* the number of men, the *larger* the number of days. When, therefore, we set down the ratio of 8 men to 12 men, and the ratio of the time required by 8 men to the time required by 12 men, we have two *un*equal ratios;

men.		men.				days.
8	:	12	|	*a*	:	14
8	:	12	::	14	:	*a*

but, by inverting* one of them, we obtain two equal ratios, which—when the equality is expressed—constitute an INVERSE Proportion.

(IV.) The breadth and the length of a piece of cloth—the quantity of cloth remaining unaltered—vary INVERSELY: the greater the breadth, the less the length; and the less the breadth, the greater the length. When, therefore, we set down the ratio of the breadth of a piece of cloth to the breadth of a narrower piece containing the same quantity, and the ratio of the length of the first piece to the length of the second, we have two *un*equal ratios; but the inversion of one of them gives us two equal ratios, which—when the equality is expressed—constitute an INVERSE Proportion.

yd.		yd.				yds.
$\frac{5}{8}$	:	$\frac{7}{12}$	|	*a*	:	30
$\frac{5}{8}$	:	$\frac{7}{12}$	::	30	:	*a*

The Unit Method.—Instead of DIRECTLY comparing the first term of a Proportion to the second term, some Arithmeticians —chiefly of the French school—compare the first term to *unity*, and then compare unity to the second term. According to this roundabout method, which is known as the "Unit Method," an ordinary question in Proportion is broken up (so to speak) into two questions; and the work is hardly ever free from fractions. No Proportion, however, is formally set down.

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* A ratio—like a fraction—is said to be "inverted" when its terms *are interchanged: the* antecedent becoming the consequent, and the *consequent becoming the* antecedent.

We now proceed to apply the Unit Method to the examples already under consideration:—

(I.) The price of 4 lbs. of tea being 18*s.*, what is the price of 1 lb.? [*Ans.* $\frac{18}{4}$*s.*] The price of 1 lb. being $\frac{18}{4}$*s.*, what is the price of 10 lbs.?

lbs. lb. *s.*	lb. lbs. *s.*
$4 : 1 :: 18 : a$	$1 : 10 :: \frac{18}{4} : a$
$a = \frac{18}{4}s.$	$a = \frac{18}{4}s. \times 10 = \frac{180}{4}s. = 45s.$

(II.) The rent of 3 acres of land being £5 13*s.* 6*d.*, or 1362*d.*, what is the rent of 1 perch? [*Ans.* $\frac{1362}{480}$*d.*] The rent of 1 perch being $\frac{1362}{480}$*d.*, what is the rent of 29A. 1R. 17P., or of 4,697 perches?

per. per. *d.*	per. per. *d.*
$480 : 1 :: 1362 : a$	$1 : 4697 :: \frac{1362}{480} : a$
$a = \frac{1362}{480}d.$	$a = \frac{1362}{480}d. \times 4697 = \frac{6397314}{480}d.$ $= 13328d.$, or £55 10*s.* 8*d.* (nearly)

(III.) If 12 men can reap a field of corn in 14 days, in what time could 1 man reap it? [*Ans.* 14×12 days.] If 1 man could reap the field in 14×12 days, in what time could 8 men reap it?

man. men. days.	men. man. days.
$1 : 12 :: 14 : a$	$8 : 1 :: 14 \times 12 : a$
$a = 14 \times 12$ days.	$a = \frac{14 \times 12}{8}$ days $= 21$ days.

(IV.) A piece of cloth being 30 yards long and $\frac{7}{12}$ of a yard broad, what is the length of another piece 1 yard broad, and containing the same quantity of cloth? [*Ans.* $\frac{30 \times 7}{12}$ yds.] A piece of cloth being $\frac{30 \times 7}{12}$ yards long and 1 yard broad, what is the length of another piece $\frac{5}{8}$ of a yard broad, and containing the same quantity of cloth?

yd. yd. yds.	yd. yd. yds.
$1 : \frac{7}{12} :: 30 : a$	$\frac{5}{8} : 1 :: \frac{30 \times 7}{12} : a$
$a = \frac{30 \times 7}{12}$ yds.	$a = \frac{30 \times 7}{12} \div \frac{5}{8} = \frac{30 \times 7}{12} \times \frac{8}{5} = \frac{30 \times 7 \times 8}{12 \times 5} = 28$ yds.

In practice, the work would assume this form:—

(I.)

lbs.		s.
4	cost	18
1	,,	$\frac{18}{4}$
10	,,	$\frac{18}{4}\times 10$
		$=45$

(II.)

$$\begin{array}{rl} & \text{per.} \quad d. \\ \text{Rent of } 480 &= 1362 \\ ,, \quad 1 &= \frac{1362}{480} \\ ,, \quad 4697 &= \frac{1362}{480}\times 4697 \\ &= \frac{6397314}{480} \\ &= 13328 \end{array}$$

or £55 10*s*. 8*d*. (nearly)

(III.)

$$\begin{array}{rl} & \text{men.} \quad \text{days.} \\ \text{Time required by } 12 &= 14 \\ ,, \quad ,, \quad ,, \quad 1 &= 14\times 12 \\ ,, \quad ,, \quad ,, \quad 8 &= \frac{14\times 12}{8} = \\ & 21 \end{array}$$

(IV.)

$$\begin{array}{rl} \text{yd.} & \text{yds.} \\ \text{Breadth being } \frac{7}{12}, \text{ length} &= 30 \\ ,, \quad ,, \quad 1, \quad ,, &= \frac{30\times 7}{12} \\ ,, \quad ,, \quad \frac{5}{8}, \quad ,, &= \frac{30\times 7\times 8}{12\times 5} \\ &= 28 \end{array}$$

Or thus:

$$\begin{array}{rl} \text{yd.} & \text{yds.} \\ \text{Breadth being } \frac{7}{12}, \text{ length} &= 30 \\ ,, \quad ,, \quad \frac{1}{12}, \quad ,, &= 30\times 7 \\ ,, \quad ,, \quad 1, \quad ,, &= \frac{30\times 7}{12} \\ ,, \quad ,, \quad \frac{1}{8}, \quad ,, &= \frac{30\times 7\times 8}{12} \\ ,, \quad ,, \quad \frac{5}{8}, \quad ,, &= \frac{30\times 7\times 8}{12\times 5} \end{array}$$

Transformations of which a Proportion is susceptible.—A Proportion is susceptible of a great variety of transformations, but it will be sufficient for our purpose to notice a few of the more important. The transformation which a Proportion undergoes is called

(I.) Inversion when—both ratios being "inverted"—we say *as the second term is to the first, so is the fourth term to the third;*

(II.) Alternation* when—the four terms being taken "alternately"—we say *as the first term is to the third, so is the second term to the fourth;*

(III.) Composition when we say *as the sum of the first and second terms is to the second, so is the sum of the third and fourth terms to the fourth;*

---

* It is hardly necessary to observe that Alternation can be employed only when the terms of the Proportion are *abstract* numbers, or when the whole four are of the *same kind.* The Proportion 4 lbs. : 10 lbs. : : 18*s*. : 45*s*., for instance, is obviously one to which Alternation is not applicable; because it would be absurd to say 4 lbs. : 18s. : : 10 lbs. : 45*s*.

(IV.) DIVISION when we say *as the excess of the first term above the second is to the second, so is the excess of the third term above the fourth to the fourth ;*

&c. Thus, from the Proportion

	$11 : 7 ::$	$33 : 21$ we obtain, by
(I.) INVERSION,	$7 : 11 ::$	$21 : 33$
(II.) ALTERNATION,	$11 : 33 ::$	$7 : 21$
(III.) COMPOSITION,	$11+7 : 7 ::$	$33+21 : 21$
(IV.) DIVISION,	$11-7 : 7 ::$	$33-21 : 21$
&c.		&c.

These transformations are easily understood :

(I.) The product of the means being equal to the product of the extremes, we have, from the original Proportion, $7\times33=11\times21$; (dividing by 11×33) $\frac{7\times33}{11\times33}=\frac{11\times21}{11\times33}$; (dividing the terms of the first fraction by 33, and those of the second by 11) $\frac{7}{11}=\frac{21}{33}$; and therefore $7 : 11 :: 21 : 33$.

$$11 : 7 :: 33 : 21$$
$$7\times33=11\times21$$
$$\frac{7\times33}{11\times33}=\frac{11\times21}{11\times33}$$
$$\frac{7}{11}=\frac{21}{33}$$
$$7 : 11 :: 21 : 33$$

(II.) The product of the extremes being equal to the product of the means, we have, from the original Proportion, $11\times21=7\times33$; (dividing by 33×21) $\frac{11\times21}{33\times21}=\frac{7\times33}{33\times21}$; (dividing the terms of the first fraction by 21, and those of the second by 33) $\frac{11}{33}=\frac{7}{21}$; and therefore $11 : 33 :: 7 : 21$.

$$11 : 7 :: 33 : 21$$
$$11\times21=7\times33$$
$$\frac{11\times21}{33\times21}=\frac{7\times33}{33\times21}$$
$$\frac{11}{33}=\frac{7}{21}$$
$$11 : 33 :: 7 : 21$$

(III. and IV.) From the original Proportion we have $\frac{11}{7}=\frac{33}{21}$; and the fractions $\frac{7}{7}$ and $\frac{21}{21}$ are equal—each being equivalent to unity. Adding equals to equals, therefore, we have (III.) $\frac{11}{7}+\frac{7}{7}=\frac{33}{21}+\frac{21}{21}$, or $\frac{11+7}{7}=\frac{33+21}{21}$; and, taking equals from equals, *(IV.)* $\frac{11}{7}-\frac{7}{7}=\frac{33}{21}-\frac{21}{21}$, or $\frac{11-7}{7}=\frac{33-21}{21}$. We thus

$$11 : 7 :: 33 : 21$$
$$\begin{cases}\frac{11}{7}=\frac{33}{21}\\ \frac{7}{7}=\frac{21}{21}\end{cases}$$
$$\text{(III.)}\ \frac{11+7}{7}=\frac{33+21}{21}$$
$$\text{(IV.)}\ \frac{11-7}{7}=\frac{33-21}{21}$$
$$\text{(III.)}\ 11+7 : 7 :: 33+21 : 21$$
$$\text{(IV.)}\ 11-7 : 7 :: 33-21 : 21$$

obtain the Proportions (III.) $11+7 : 7 :: 33+21 : 21$, and (IV.) $11-7 : 7 :: 33-21 : 21$.*

167. The first of four numbers bears to the second the same ratio which the third bears to the fourth, when the product of the first by the fourth is equal to the product of the second by the third.

Taking, for example, the four numbers

$$11, \quad 7, \quad 33, \quad \text{and } 21,$$

and finding $11\times21=7\times33$, we have (dividing by 7×21) $\frac{11\times21}{7\times21}=\frac{7\times33}{7\times21}$; (dividing the terms of the first fraction by 21, and those of the second by 7) $\frac{11}{7}=\frac{33}{21}$; and therefore

$$11 : 7 :: 33 : 21\dagger$$

168. We obtain a Proportion from an exercise in Multiplication by saying—as unity is to the multiplier, so is the multiplicand to the product; and from an exercise in Division, *when there is no "remainder,"* by saying—as the divisor is to unity, so is the dividend to the quotient.

---

* If, employing general symbols, we take the Proportion $a : b :: c : d$, we shall have—

(I.)

$$a : b :: c : d$$
$$b\times c=a\times d$$
$$\frac{b\times c}{a\times c}=\frac{a\times d}{a\times c}$$
$$\frac{b}{a}=\frac{d}{c}$$
$$b : a :: d : c$$

(II.)

$$a : b :: c : d$$
$$a\times d=b\times c$$
$$\frac{a\times d}{c\times d}=\frac{b\times c}{c\times d}$$
$$\frac{a}{c}=\frac{b}{d}$$
$$a : c :: b : d$$

(III. and IV.)

$$a : b :: c : d$$
$$\begin{cases} \frac{a}{b}=\frac{c}{d} \\ \frac{b}{b}=\frac{d}{d} \end{cases}$$

(III.) $\frac{a+b}{b}=\frac{c+d}{d}$

(IV.) $\frac{a-b}{b}=\frac{c-d}{d}$

(III.) $a+b : b :: c+d : d$
(IV.) $a-b : b :: c-d : d$

† Taking the general symbols

$$a, \quad b, \quad c, \quad \text{and } d,$$

and finding $a\times d=b\times c$, we have $\frac{a\times d}{b\times d}=\frac{b\times c}{b\times d}$; $\frac{a}{b}=\frac{c}{d}$; and therefore

$$a : b :: c : d$$

Thus, from the exercise

		we obtain the Proportion
Multiplication.	(*a*) $19 \times 8 = 152$,	(*a*) $1 : 8 :: 19 : 152$
	(*b*) $23 \times \cdot 7 = 16\cdot 1$,	(*b*) $1 : \cdot 7 :: 23 : 16\cdot 1$
	(*c*) 4*s.* 7*d.* × 9 = £2 1*s.* 3*d.*	(*c*) 1 : 9 :: 4*s.* 7*d.* : £2 1*s.* 3*d.*
	(*d*) $\frac{5}{7} \times \frac{2}{3} = \frac{10}{21}$,	(*d*) $1 : \frac{2}{3} :: \frac{5}{7} : \frac{10}{21}$
Division.	(*e*) $152 \div 8 = 19$,	(*e*) $8 : 1 :: 152 : 19$
	(*f*) $16\cdot 1 \div \cdot 7 = 23$,	(*f*) $\cdot 7 : 1 :: 16\cdot 1 : 23$
	(*g*) £2 1*s.* 3*d.* ÷ 9 = 4*s.* 7*d.*	(*g*) 9 : 1 :: £2 1*s.* 3*d.* : 4*s.* 7*d.*
	(*h*) $\frac{10}{21} \div \frac{2}{3} = \frac{5}{7}$,	(*h*) $\frac{2}{3} : 1 :: \frac{10}{21} : \frac{5}{7}$
	&c.	&c.

This follows directly from § 167 : because in every such case as the preceding ones, the product of the first number by the fourth is necessarily equal to the product of the second by the third.

The connexion—just noticed—between Multiplication and Division on the one hand, and Proportion on the other, suggests a passing allusion to the following very old definitions:

(I.) Multiplication is an operation by which, when two numbers [the "multiplicand" and the "multiplier"] are given, we obtain a third [the "product"]—derived from one of the given numbers [the "multiplicand"] in the same way as the other [the "multiplier"] is derived from unity.

Thus, to obtain—

(*a*) 8 from 1, we multiply by 8;	152 from 19, we multiply by 8.
(*b*) ·7 from 1, we multiply by 7 and divide by 10;	16·1 from 23, we multiply by 7 and divide by 10.
(*c*) 9 from 1, we multiply by 9;	£2 1*s.* 3*d.* from 4*s.* 7*d.*, we multiply by 9.
(*d*) $\frac{2}{3}$ from 1, we multiply by 2 and divide by 3;	$\frac{10}{21}$ from $\frac{5}{7}$, we multiply by 2 and divide by 3.

(II.) Division is an operation by which, when two numbers [the "dividend" and the "divisor"] are given, we obtain a third [the "quotient"]—derived from one of the given numbers [the "dividend"] in the same way as unity is derived from the other [the "divisor"].

Thus, to obtain—

(*e*) 1 from 8, we divide by 8;	19 from 152, we divide by 8.
(*f*) 1 from ·7, we multiply by 10 and divide by 7;	23 from 16·1, we multiply by 10 and divide by 7.
(*g*) 1 from 9, we divide by 9;	4*s.* 7*d.* from £2 1*s.* 3*d.*, we divide by 9.
(*h*) 1 from $\frac{2}{3}$, we multiply by 3 and divide by 2;	$\frac{5}{7}$ from $\frac{10}{21}$, we multiply by 3 and divide by 2.

Looked at from an Arithmetical point of view, the preceding definitions, although ingenious and plausible, are by no means unexceptionable. The first assumes that a multiplier is derivable from unity in only one way. The second definition assumes that unity is derivable from a divisor in only one way; also, that a "remainder" never occurs in an exercise in Division.

Now, everybody knows that a multiplier is derivable from unity, and that unity is derivable from a divisor, in more ways than one; so that the definitions hold good only in a particular sense. For instance:

(I.)

(*a*) We obtain 8 from 1 by adding 7;	we do *not* obtain 152 from 19 by adding 7.
(*b*) We obtain ·7 from 1 by subtracting ·3;	we do *not* obtain 16·1 from 23 by subtracting ·3.
(*c*) We obtain 9 from 1 by adding 8.	we do *not* obtain £2 1*s*. 3*d*. from 4*s*. 7*d*. by adding 8.
(*d*) We obtain $\frac{2}{3}$ from 1 by subtracting $\frac{1}{3}$;	we do *not* obtain $\frac{10}{21}$ from 5 by subtracting $\frac{1}{3}$.

(II.)

(*e*) We obtain 1 rom 8 by subtracting 7;	we do *not* obtain 19 from 152 by subtracting 7.
(*f*) We obtain 1 from ·7 by adding ·3;	we do *not* obtain 23 from 16·1 by adding ·3.
(*g*) We obtain 1 from 9 by subtracting 8;	we do *not* obtain 4*s*. 7*d*. from £2 1*s*. 3*d*. by subtracting 8.
(*h*) We obtain 1 from $\frac{2}{3}$ by adding $\frac{1}{3}$;	we do *not* obtain $\frac{5}{7}$ from $\frac{10}{21}$ by adding $\frac{1}{3}$.

COMPOUND PROPORTION.*

169. When two or more ratios are given, and we multiply the antecedents together for a new antecedent, and the consequents together for a new consequent, the ratio which the new antecedent bears to the new consequent is said to be COMPOUNDED of the two or more given ratios.

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* Compound Proportion is sometimes called the "DOUBLE Rule of Three:" a name which, although long in use, can hardly be considered appropriate—particularly when there are *more than two* ratios to be compounded.

Thus, the ratio $3 \times 6 : 4 \times 5$ is "compounded" of the two ratios $3:4$ and $6:5$; the ratio $9 \times 2 \times 11 : 8 \times 3 \times 7$ is "compounded" of the three ratios $9:8$, $2:3$, and $11:7$; &c.

$$\begin{array}{ccc|ccc} & & & 9 & : & 8 \\ 3 & : & 4 & 2 & : & 3 \\ 6 & : & 5 & 11 & : & 7 \\ \hline 3\times 6 & : & 4\times 5 & 9\times 2\times 11 & : & 8\times 3\times 7 \end{array}$$

170. A Proportion is called COMPOUND when one of the ratios—instead of being expressed in the ordinary way—is represented by two or more other ratios, from the "compounding" of which it is ultimately obtained.

The following are "Compound" Proportions:—

(I.)

$$\left.\begin{matrix} 3:4 \\ 6:5 \end{matrix}\right\} :: 9:10$$

(II.)

$$\left.\begin{matrix} 9:8 \\ 2:3 \\ 11:7 \end{matrix}\right\} :: 33:28.$$

The first of these Proportions indicates that the ratio $9:10$ is equal—not to either $3:4$ or $6:5$, but—to the ratio $(18:20)$ *compounded* of $3:4$ and $6:5$. The second Proportion indicates that the ratio $33:28$ is equal—not to $9:8$, or $2:3$, or $11:7$, but—to the ratio $(198:168)$ *compounded* of $9:8$, $2:3$, and $11:7$.

171. When the ratio represented by two or more others is obtained from the "compounding" of those others, and substituted for them, a COMPOUND Proportion becomes a SIMPLE Proportion.

Thus, the Compound Proportion—

$$\left.\begin{matrix} \left.\begin{matrix} 3:4 \\ 6:5 \end{matrix}\right\} :: 9:10 \\ \\ \left.\begin{matrix} 9:8 \\ 2:3 \\ 11:7 \end{matrix}\right\} :: 33:28 \end{matrix}\right\} \text{becomes a Simple Proportion when written} \left\{\begin{matrix} 18:20::9:10 \\ \\ 198:168::33:28 \end{matrix}\right.$$

PRACTICAL EXERCISES IN COMPOUND PROPORTION.

EXAMPLE I.—If 8 horses can plough 20 acres in 7 days, how many acres could 5 horses plough in 11 days?

H.	A.	DYS.
8	20	7
5	?	11

Here we have materials for two exercises in Simple Proportion:

(1) If 8 horses can plough 20 acres in a certain time [7 days], how many acres could 5 horses plough in the *same* time?

$$\overset{\text{H.}}{8} : \overset{\text{H.}}{5} :: \overset{\text{A.}}{20} : a;\ a = \frac{20 \times 5}{8} \text{ acres.}$$

(2) If a certain number of horses [5] can plough $\frac{20 \times 5}{8}$ acres in 7 days, how many acres could the *same* number of horses plough in 11 days?

$$\overset{\text{DYS.}}{7} : \overset{\text{DYS.}}{11} :: \overset{\text{A.}}{\frac{20 \times 5}{8}} : a;\ a = \frac{20 \times 5 \times 11}{8 \times 7} \text{ acres.}$$

By means of these Proportions we find (1) that 5 horses, in 7 days, could plough $\frac{20 \times 5}{8}$ acres; and (2) that 5 horses, in 11 days, could plough $\frac{20 \times 5 \times 11}{8 \times 7}$ acres.

It will be seen that the ratio 20 acres : $\frac{20 \times 5 \times 11}{8 \times 7}$ acres is equal —not to either 8 horses : 5 horses, or 7 days : 11 days, but—to the ratio ($8 \times 7 : 5 \times 11$) *compounded* of 8 : 5 and 7 : 11. It will likewise be seen (1) that, when writing 8 horses in the first, and 5 horses in the second place, we assumed the number of days to be the same in both cases; and (2) that, when writing 7 days in the first, and 11 days in the second place, we assumed the number of horses to be the same in both cases. In practice, the work would stand thus:—

$$\left.\begin{array}{ccc} 8 & : & 5 \\ 7 & : & 11 \end{array}\right\} :: 20 : a$$

$$8 \times 7 : 5 \times 11 :: 20 : a$$

$$a = 20 \times 5 \times 11 \div 8 \times 7 = 1100 \div 56 = 19\tfrac{36}{56} \text{ or } 19\tfrac{9}{14} \text{ acres.}$$

EXAMPLE II.—If 6 men, working 10 hours a day, can build a wall 360 yards long in 15 days, in what time could 4 men, working 12 hours a day, build a length of 480 yards?

M.	HRS.	YDS.	DYS.
6	10	360	15
4	12	480	?

Here we have materials for three exercises in Simple Proportion:

(1) If 6 men can do a certain quantity of work [360 yards] in 15 days, in how many days of the *same* length [10 hrs.] could 4 men do the *same* quantity of work?

$$\overset{\text{M.}}{4} : \overset{\text{M.}}{6} :: \overset{\text{DYS.}}{15} : a;\ a = \frac{15 \times 6}{4} \text{ days.}$$

(2) If a number of men [4], engaged 10 hours a day, can do a certain quantity of work [360 yds.] in $\frac{15\times 6}{4}$ days, in what time could the *same* number of men, engaged 12 hours a day, do the *same* quantity of work?

$$\begin{array}{ccccc} \text{HRS.} & & \text{HRS.} & & \text{dys.} \\ 12 & : & 10 & : & \frac{15\times 6}{4} \end{array} : a;\ a = \frac{15\times 6\times 10}{4\times 12} \text{ days.}$$

(3) If a certain number of men [4], working a certain number of hours [12] daily, can build 360 yards of a wall in $\frac{15\times 6\times 10}{4\times 12}$ days, in what time could the *same* number of men, working the *same* number of hours daily, build 480 yards?

$$\begin{array}{ccccc} \text{YDS.} & & \text{YDS.} & & \text{dys.} \\ 360 & : & 480 & :: & \frac{15\times 6\times 10}{4\times 12} \end{array} : a;\ a = \frac{15\times 6\times 10\times 480}{4\times 12\times 360} \text{ days.}$$

By means of these Proportions we find (1) that 4 men, working 10 hours a day, could build 360 yards in $\frac{15\times 6}{4}$ days; (2) that 4 men, working 12 hours a day, could build 360 yards in $\frac{15\times 6\times 10}{4\times 12}$ days; and (3) that 4 men, working 12 hours a day, could build 480 yards in $\frac{15\times 6\times 10\times 480}{4\times 12\times 360}$ days.

It will be observed that the ratio 15 days : $\frac{15\times 6\times 10\times 480}{4\times 12\times 360}$ days is equal—not to 4 men : 6 men, or 12 hours : 10 hours, or 360 yards : 480 yards, but—to the ratio ($4\times 12\times 360 : 6\times 10\times 480$) *compounded* of 4 : 6, 12 : 10, and 360 : 480. It will also be observed (1) that, when writing 4 men in the first, and 6 men in the second place, we supposed the length of the day, as well as the quantity of work, to be the same in both cases; (2) that, when writing 12 hours in the first, and 10 hours in the second place, we supposed the number of men, as well as the quantity of work, to be the same in both cases; and (3) that, when writing 360 yards in the first, and 480 yards in the second place, we supposed the number of men, as well as the length of the day, to be the same in both cases. Here is the work, as it would appear in practice:—

$$\left.\begin{array}{r} 4 : 6 \\ 12 : 10 \\ 360 : 480 \end{array}\right\} :: \quad 15 : a.$$

$$4\times 12\times 360 : 6\times 10\times 480 :: 15 : a.$$

$$a = 15\times 6\times 10\times 480 \div 4\times 12\times 360 = 432000 \div 17280 = 25 \text{ days.}$$

EXAMPLE III.—If 10 weavers, in 5 days, can make 30 pieces of cloth, each 16 feet long and 2 feet broad, how many pieces, each 12 feet long and 3 feet broad, could be made by 9 weavers in 8 days?

W.	dys.	P.	long.	broad.
10	5	30	16 ft.	2 ft.
9	8	?	12 ,,	3 ,,

Leaving the fourth place for *a* (the "answer"), and making 30 pieces the third term, we know, after what has just been explained, that the ratio 30 : *a* is equal to one "compounded" of four other ratios—namely:

(1)	either	10 : 9	or	9 : 10
(2)	,,	5 : 8	,,	8 : 5
(3)	,,	16 : 12	,,	12 : 16
(4)	,,	2 : 3	,,	3 : 2

In determining, in the case of each of the four ratios referred to, which term should be made the antecedent, and which the consequent, we have merely to take, in turn, every two numbers of the same kind, and (disregarding, for the moment, the remaining three pairs) consider how they should be placed if they and the third term were the *only* numbers upon which the answer depended:—

(1) If 10 weavers can make 30 pieces of cloth under certain circumstances, how many such pieces could 9 weavers make under the *same* circumstances? The Proportion being 10 : 9 : : 30 : *a*, we write 10 in the first, and 9 in the second place.

(2) If 30 pieces of cloth can, under certain circumstances, be made in 5 days, how many such pieces could, under the *same* circumstances, be made in 8 days? The Proportion being 5 : 8 : : 30 : *a*, we write 5 in the first, and 8 in the second place.

(3) If 30 pieces of cloth, each 16 feet long, can be made under certain circumstances, how many pieces (of the same breadth), each 12 feet long, could be made under the *same* circumstances? The Proportion being 12 : 16 : : 30 : *a*, we write 12 in the first, and 16 in the second place.

(4) If 30 pieces of cloth, each 2 feet broad, can be made under certain circumstances, how many pieces (of the same length), each 3 feet broad, could be made under the *same* circumstances? The proportion being 3 : 2 : : 30 : *a*, we write 3 in the first, and 2 in the second place.

We thus obtain the Compound Proportion—

$$\left.\begin{array}{r} 10:9 \\ 5:8 \\ 12:16 \\ 3:2 \end{array}\right\} :: 30:a.$$

And from the resulting Simple Proportion,

$$10\times5\times12\times3 : 9\times8\times16\times2 :: 30 : a,$$

we find $a=30\times9\times8\times16\times2\div10\times5\times12\times3=69,120\div 1,800=38\frac{2}{5}$ pieces.

172. Rule for the working of exercises in Compound Proportion: Leaving the fourth place for the answer, write, as the third term, the number which is of the same kind as the answer. Of the remaining numbers, take any two which are of the same kind, and (having, if necessary, reduced them to the same denomination) set them down—one in the first, the other in the second place—as if they and the third term were the *only* numbers upon which the answer depended. Treat in a similar manner every other pair of numbers of the same kind. Then say—as the product of the numbers in the first place is to the product of those in the second place, so is the third term to the answer; and from the Simple Proportion so obtained, find the answer in the ordinary way (§ 166).

Here are the solutions of the preceding examples according to the Unit Method—

(I.)

H.		D.			A.
8	in	7	can plough		20
1	„	7	„	„	$\frac{20}{8}$
1	„	1	„	„	$\frac{20}{8\times7}$
5	„	1	„	„	$\frac{20\times5}{8\times7}$
5	„	11	„	„	$\frac{20\times5\times11}{8\times7}$

(II.)

M.		H.			Y.		D.
6	working	10	daily	can build	360	in	15
1	”	10	”	”	360	”	15×6
1	”	1	”	”	360	”	$15\times6\times10$
1	”	1	”	”	1	”	$\frac{15\times6\times10}{360}$
4	”	1	”	”	1	”	$\frac{15\times6\times10}{360\times4}$
4	”	12	”	”	1	”	$\frac{15\times6\times10}{360\times4\times12}$
4	”	12	”	”	480	”	$\frac{15\times6\times10\times480}{360\times4\times12}$

(III.)

W.		D.			lg.			bd.			P.
10	in	5	can,	of cloth	16	ft.	and	2	ft.,	make	30
1	”	5	”	”	16	”	”	2	”	”	$\frac{30}{10}$
1	”	1	”	”	16	”	”	2	”	”	$\frac{30}{10\times5}$
1	”	1	”	”	1	”	”	2	”	”	$\frac{30\times16}{10\times5}$
1	”	1	”	”	1	”	”	1	”	”	$\frac{30\times16\times2}{10\times5}$
9	”	1	”	”	1	”	”	1	”	”	$\frac{30\times16\times2\times9}{10\times5}$
9	”	8	”	”	1	”	”	1	”	”	$\frac{30\times16\times2\times9\times8}{10\times5}$
9	”	8	”	”	12	”	”	1	”	”	$\frac{30\times16\times2\times9\times8}{10\times5\times12}$
9	”	8	”	”	12	”	”	3	”	”	$\frac{30\times16\times2\times9\times8}{10\times5\times12\times3}$

The Chain Rule.—Certain exercises in Compound Proportion can be worked with great facility by what is called the "Chain Rule," which will be understood from the following illustration:—

If 5 lbs. of tea be worth 9 lbs. of coffee, and 4 lbs. of coffee worth 17 lbs. of sugar, and 8 lbs. of sugar worth 3 lbs. of butter, and 2 lbs. of butter worth 7 lbs. of rice, what quantity of rice is worth 12 lbs. of tea?

Arranging the numbers in the way shown in the margin, we simply divide the product $(9 \times 17 \times 3 \times 7 \times 12)$ of those in the second column by the product $(5 \times 4 \times 8 \times 2)$ of those in the first column. The fact can easily be established that the resulting quotient is the answer.

lbs.		lbs.
5 tea	=	9 coffee
4 coffee	=	17 sugar
8 sugar	=	3 butter
2 butter	=	7 rice
? rice	=	12 tea

$$a = 9 \times 17 \times 3 \times 7 \times 12 \div 5 \times 4 \times 8 \times 2$$
$$= 38{,}556 \div 320 = 120\tfrac{156}{320} \text{ lbs. rice.}$$

(I.) If 5 lbs. of tea be worth 9 lbs. of coffee, 12 lbs. of tea must be worth $\frac{12 \times 9}{5}$ lbs. of coffee:

$$\begin{array}{ccccccc} \text{t.} & & \text{t.} & & \text{c.} & & \text{c.} \\ 5 & : & 12 & :: & 9 & : & \frac{12 \times 9}{5}. \end{array}$$

(II.) If 4 lbs. of coffee be worth 17 lbs. of sugar, $\frac{12 \times 9}{5}$ lbs. of coffee must be worth $\frac{12 \times 9 \times 17}{5 \times 4}$ lbs. of sugar:

$$\begin{array}{ccccccc} \text{c.} & & \text{c.} & & \text{s.} & & \text{s.} \\ 4 & : & \frac{12 \times 9}{5} & :: & 17 & : & \frac{12 \times 9 \times 17}{5 \times 4}. \end{array}$$

(III.) If 8 lbs. of sugar be worth 3 lbs. of butter, $\frac{12 \times 9 \times 17}{5 \times 4}$ lbs. of sugar must be worth $\frac{12 \times 9 \times 17 \times 3}{5 \times 4 \times 8}$ lbs. of butter:

$$\begin{array}{ccccccc} \text{s.} & & \text{s.} & & \text{b.} & & \text{b.} \\ 8 & : & \frac{12 \times 9 \times 17}{5 \times 4} & :: & 3 & : & \frac{12 \times 9 \times 17 \times 3}{5 \times 4 \times 8}. \end{array}$$

(IV.) If 2 lbs. of butter be worth 7 lbs. of rice, $\frac{12 \times 9 \times 17 \times 3}{5 \times 4 \times 8}$ lbs. of butter must be worth $\frac{12 \times 9 \times 17 \times 3 \times 7}{5 \times 4 \times 8 \times 2}$ lbs. of rice:

$$\begin{array}{ccccccc} \text{b.} & & \text{b.} & & \text{r.} & & \text{r.} \\ 2 & : & \frac{12 \times 9 \times 17 \times 3}{5 \times 4 \times 8} & :: & 7 & : & \frac{12 \times 9 \times 17 \times 3 \times 7}{5 \times 4 \times 8 \times 2}. \end{array}$$

We thus find 12 lbs. of tea to be worth (I.) $\frac{12 \times 9}{5}$ lbs. of coffee, or (II.) $\frac{12 \times 9 \times 17}{5 \times 4}$ lbs. of sugar, or (III.) $\frac{12 \times 9 \times 17 \times 3}{5 \times 4 \times 8}$ lbs. of butter, or (IV.) $\frac{12 \times 9 \times 17 \times 3 \times 7}{5 \times 4 \times 8 \times 2}$ lbs. of rice. So that

the ratio 12 : a is compounded of the four ratios 5 : 9, 4 : 17, 8 : 3, and 2 : 7—

$$\left.\begin{array}{r} 5 : 9 \\ 4 : 17 \\ 8 : 3 \\ 2 : 7 \end{array}\right\} :: 12 : a$$

$$5\times4\times8\times2 : 9\times17\times3\times7 :: 12 : a$$

$$a = 12\times9\times17\times3\times7 \div 5\times4\times8\times2.$$

The Chain Rule is employed principally in the working of a class of exercises in EXCHANGE.

PRACTICE.

173. A *concrete* number which is contained an exact number of times in another, is said to be an ALIQUOT PART of that other.

Thus, 4*s.* is an aliquot part of 12*s.*, but not of 18*s.*; 5 cwt. is an aliquot part of 10 cwt., but not of 14 cwt.; 3 yds. is an aliquot part of 21 yds., but not of 26 yds.; &c. "Aliquot part," therefore, may be said to be synonymous with "measure," (see § 89)—the only difference being that the former expression is used when the numbers are concrete, and the latter when the numbers are abstract.

174. PRACTICE teaches us how to employ our knowledge of Fractions* in finding, by means of aliquot parts, the price of any number of articles when the price of one is given; also, the price of any particular quantity of merchandise, &c., when —as is almost invariably the case in commercial transactions—the given price is that of some *unit*.

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* So far, however, as Practice is concerned, an extensive knowledge of Fractions is by no means necessary. In the great majority of instances, the only Fractions to be dealt with are the Fractional UNITS ($\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, &c.), which, at a comparatively early stage, pupils can easily be taught to read and write—the teaching being, of course, accompanied by an explanation of what is meant by a *half*, a *third*, a *fourth*, a *fifth*, &c., of anything.

As every exercise in Practice can be worked in more ways than one, the pupil, in determining the best solution in each case, must rely upon his judgment and ingenuity, rather than upon any formal Rule which could be laid down. The following examples, however, may assist the pupil, who, in selecting "aliquot parts," will naturally employ larger divisors than 12 as seldom as possible:—

Class 1, in which the quantity whose price is required is expressed by a simple number of the same denomination as that of the unit whose price is given. Every example in this class may be regarded as belonging to some one of four sub-classes, according as the given price is (*a*) less than 1*d*., (*b*) between 1*d*. and 1*s*., (*c*) between 1*s*. and £1, or (*d*) more than £1. The instances in which the given price is 1*d*., 1*s*., or £1 need not be noticed: because the merest child can tell that, at 1*d*., 1*s*., or £1 each, any number of articles would cost that number of pence, or of shillings, or of pounds—as the case may be.

(*a*.) Example I.—Find the price of 78 apples, at $\frac{1}{2}$*d*. each.

At 1*d*. each, 78 apples would cost 78*d*.; therefore, at $\frac{1}{2}$*d*. ($\frac{1}{2}$ of 1*d*.) each, 78 apples must cost $\frac{1}{2}$ of 78*d*.—that is, (78*d*.÷2=) 39*d*., or 3*s*. 3*d*.:

78 @ $\frac{1}{2}$*d*. each.

$\frac{1}{2}$*d*.=$\frac{1}{2}$ of 1*d*.	78*d*.=price @ 1*d*. each.
	39*d*. or 3*s*. 3*d*. } " " $\frac{1}{2}$*d*. "

(*a*.) Example II.—Find the price of 234 oranges, at $\frac{3}{4}$*d*. each.

Regarding $\frac{3}{4}$*d*. as the difference between 1*d*. and $\frac{1}{4}$*d*., we see that 234 oranges would, at 1*d*. each, cost 234*d*.; at $\frac{1}{4}$*d*. ($\frac{1}{4}$ of 1*d*.) each, $\frac{1}{4}$ of 234*d*.—that is, (234*d*.÷4=) 58$\frac{1}{2}$*d*.; and at $\frac{3}{4}$*d*. each, 175$\frac{1}{2}$*d*., or 14*s*. 7$\frac{1}{2}$*d*.,* the difference between 234*d*. and 58$\frac{1}{2}$*d*.

234 @ $\frac{3}{4}$*d*. each.

	d.
$\frac{1}{4}$*d*.=$\frac{1}{4}$ of 1*d*.	234 =price @ 1*d*. each.
	58$\frac{1}{2}$= " " $\frac{1}{4}$*d*. "
	175$\frac{1}{2}$ or 14*s*. 7$\frac{1}{2}$ }= " " $\frac{3}{4}$*d*. "

* In 234*d*. (the price of 234 oranges at 1*d*. each) there would be an overcharge of $\frac{1}{4}$*d*. for each orange, or an overcharge of 58$\frac{1}{2}$*d*. for the 234 oranges.

(*b.*) Example III.—Find the price of 57 lbs. of sugar, at 4*d.* a pound.

As 4*d.* is $\frac{1}{3}$ of 1*s.*, and the price at 1*s.* a pound would be 57*s.*, the price at 4*d.* a pound must be $\frac{1}{3}$ of 57*s.*—that is, (57*s.*÷3=) 19*s.*:

57 @ 4*d.* each.

	s.
4*d.* = $\frac{1}{3}$ of 1*s.*	57 = price @ 1*s.* each.
	19 = „ „ 4*d.* „

(*b.*) Example IV.—Find the price of 146 gallons of milk, at 10½*d.* a gallon.

The given price is not, but the difference (1½*d.*) between it and 1*s.* is, an aliquot part of 1*s.* We therefore regard 10½*d.* as the difference between 1*s.* and 1½*d.* The price of 146 gallons, at 1*s.* a gallon, would be 146*s.*; and at 1½*d.* ($\frac{1}{8}$ of 1*s.*) a gallon, $\frac{1}{8}$ of 146*s.*—that is, (146*s.*÷8=) 18*s.* 3*d.*; so that the price at 10½*d.* a gallon must be 127*s.* 9*d.*, or £6 7*s.* 9*d.*—the difference between 146*s.* and 18*s.* 3*d.*:

146 @ 10½*d.* each.

	s.	*d.*	
1½*d.*=$\frac{1}{8}$ of 1*s.*	146	0=	price @ 1*s.* each.
	18	3=	„ „ 1½*d.* „
	127	9	„ „ 10½*d.* „
	or £6 7	9	

(*b.*) Example V.—Find the price of 85 yards of ribbon, at 9¾*d.* a yard.

Neither the given price, nor the difference between it and 1*s.*, being an aliquot part of 1*s.*, we break up 9¾*d.* into 6*d.* (an aliquot part of 1*s.*), 3*d.* (an aliquot part of 6*d.*), and ¾*d.* (an aliquot part of 3*d.*). Dividing 85*s.*, the price at 1*s.* a yard, by 2, we find the price at 6*d.* (½ of 1*s.*) a yard to be 42*s.* 6*d.*; dividing this amount by 2, we find the price at 3*d.* (½ of 6*d.*) a yard to be 21*s.* 3*d.*; dividing this last amount by 4, we find the price at ¾*d.* (¼ of 3*d.*) a yard to be 5*s.* 3¾*d.*; and, adding together the price at 6*d.*, the price at 3*d.*, and the price at ¾*d.* a yard, we find the price at (6*d.*+3*d.*+¾*d.*=) 9¾*d.* a yard to be (42*s.* 6*d.*+21*s.* 3*d.*+5*s.* 3¾*d.*=) 69*s.* 0¾*d.*, or £3 9*s.* 0¾*d.*

85 @ $9\frac{3}{4}d.$ each.

d.	*s.*	*d.*	
$6 = \frac{1}{2}$ of 1*s.*	85	0	= price @ 1*s.* each.
$3 = \frac{1}{2}$ " 6*d.*	42	6	= " " 6*d.* "
$\frac{3}{4} = \frac{1}{4}$ " 3*d.*	21	3	= " " 3*d.* "
	5	$3\frac{3}{4}$	= " " $\frac{3}{4}d.$ "
	69	$0\frac{3}{4}$	
or	£3 9	$0\frac{3}{4}$	= " " $9\frac{3}{4}d.$ "

This result is more easily obtained when—regarding $9\frac{3}{4}d.$ as the difference between 10*d.* and $\frac{1}{4}d.$—we subtract the price at $\frac{1}{4}d.$ a yard from the price at 10*d.* a yard:

	d.		*d.*
Price of 85 yds. @	10	a yard $= 85 \times 10d. =$	850
	$\frac{1}{4}$	" $= 85 \times \frac{1}{4}d. =$	$21\frac{1}{4}$
	$9\frac{3}{4}$	" $=$	$828\frac{3}{4} = £3\ 9s.\ 0\frac{3}{4}d.$

(*c.*) Example VI.—Find the price of 92 lambs, at 14*s.* each.

Seeing that the given price is an *even* number of shillings, we take half this number for multiplier, and 92 for multiplicand. The resulting product, when the most right-handed figure is "cut off," expresses the *pounds* (£64) of the answer; and by doubling the cut-off figure (4) we obtain the *shillings* (8*s.*) of the answer.

92 @ 14*s.* each.

$$\begin{array}{r} 92 \\ 7 \\ \hline £64\text{-}4 \\ 8s. \end{array}$$

This is easily explained. Expressed as a fraction of £1, the given price is $£\frac{14}{20}$, or $£\frac{7}{10}$. Now, at £1 each, 92 lambs would cost £92; therefore, at $\frac{7}{10}$ of £1 each, the price must be $\frac{7}{10}$ of £92—that is, $£\frac{92 \times 7}{10} = £\frac{92}{10} \times 7 = £9{\cdot}2 \times 7 = £64{\cdot}4$ = (§ 88) £64 8*s.*

(*c.*) Example VII.—Find the price of 69 articles, at 2*s.* 6*d.* each.

As 69 articles at £1 each would cost £69, and 2*s.* 6*d.* is $\frac{1}{8}$ of £1, the price at 2*s.* 6*d.* each must be $\frac{1}{8}$ of £69—that is, (£69 ÷ 8 =) £8 12*s* 6*d.*:

69 @ 2*s.* 6*d.* each.

	£	*s.*	*d.*	
2*s.* 6*d.* $= \frac{1}{8}$ of £1	69	0	0	= price @ £1 each.
	8	12	6	= " " 2*s* 6*d.* "

(c.) Example VIII.—Find the price of 392 articles, at 16s. 8d. each.

The given price is not, but the difference (3s. 4d.) between it and £1 is, an aliquot part of £1. We therefore regard 16s. 8d. as the difference between £1 and 3s. 4d. The price of 392 articles at £1 each would be £392; and at 3s. 4d. ($\frac{1}{6}$ of £1) each, $\frac{1}{6}$ of £392—that is, (£392÷6=) £65 6s. 8d.; so that, at 16s. 8d. each, the price must be £326 13s. 4d.—the difference between the price at £1 each and the price at 3s. 4d. each:

392 @ 16s. 8d. each.

	£	s.	d.	
3s. 4d. = $\frac{1}{6}$ of £1.	392	0	0	= price @ £1 each.
	65	6	8	= ,, ,, 3s. 4d. ,,
	326	13	4	= ,, ,, 16s. 8d. ,,

(c.) Example IX.—Find the price of 108 articles, at 13s. 7¼d. each.

Neither the given price, nor the difference between it and £1, being an aliquot part of £1, we regard 13s. 7¼d. as the sum of 10s. (an aliquot part of £1), 2s. (an aliquot part of 10s.), 1s. (an aliquot part of 2s.), 6d. (an aliquot part of 1s.), 1d. (an aliquot part of 6d.), and ¼d. (an aliquot part of 1d.).

Price of 108 articles at £1 each=£108; at 10s. (½ of £1) each=½ of £108=£54; at 2s. ($\frac{1}{5}$ of 10s.) each=$\frac{1}{5}$ of £54=£10 16s.; at 1s. (½ of 2s.) each=½ of £10 16s.=£5 8s.; at 6d. (½ of 1s.) each=½ of £5 8s.=£2 14s.; at 1d. ($\frac{1}{6}$ of 6d.) each=$\frac{1}{6}$ of £2 14s.=9s.; at ¼d. (¼ of 1d.) each=¼ of 9s.=2s. 3d.; and at (10s.+2s.+1s.+6d.+1d.+¼d.=) 13s. 7¼d. each=£54+£10 16s.+£5 8s.+£2 14s.+9s.+2s. 3d.=£73 9s. 3d.:

108 @ 13s. 7¼d. each.

	£	s.	d.	
10s. = ½ of £1	108	0	0	= price @ £1 each.
2s. = $\frac{1}{5}$,, 10s.	54	0	0	= ,, ,, 10s. ,,
1s. = ½ ,, 2s.	10	16	0	= ,, ,, 2s. ,,
6d. = ½ ,, 1s.	5	8	0	= ,, ,, 1s. ,,
1d. = $\frac{1}{6}$,, 6d.	2	14	0	= ,, ,, 6d. ,,
¼d. = ¼ ,, 1d.	0	9	0	= ,, ,, 1d. ,,
	0	2	3	= ,, ,, ¼d. ,,
	73	9	3	= ,, ,, 13s. 7¼d. each.

Or thus: Price of 108 articles at 1*s.* each=108*s.*; at 13*s.* each=108*s.*×13=1,404*s.*; at 6*d.* (½ of 1*s.*) each=½ of 108*s.* =54*s.*; at 1*d.* (⅙ of 6*d.*) each=⅙ of 54*s.*=9*s.*; at ¼*d.* (¼ of 1*d.*) each=¼ of 9*s.*=2*s.* 3*d.*; and at (13*s.*+6*d.*+1*d.*+¼*d.*=) 13*s.* 7¼*d.* each=1,404*s.*+54*s.*+9*s.*+2*s.* 3*d.*=1,469*s.* 3*d.*, or £73 9*s.* 3*d.*:

	s.		
6*d.*=½ of 1*s.*	108	= price @ 1*s.* each.	
	13		
	324		
	108		
	1404	= „ „ 13*s.* „	
1*d.*=⅙ of 6*d.*	54	= „ „ 6*d.* „	
¼*d.*=¼ „ 1*d.*	9	= „ „ 1*d.* „	
	2 3*d.*	= „ „ ¼*d.* „	
	1,469 3*d.* or £73 9*s.* 3*d.*	= „ „ 13*s.* 7¼*d.* each.	

The answer is most easily obtained, however, when—taking advantage of the circumstance that 108 is resolvable into the factors 12 and 9—we simply multiply 13*s.* 7¼*d.* by 108. In performing the multiplication, we naturally begin with the factor 12, in order to get rid of the farthing.

£	*s.*	*d.*
0	13	7¼
		12
8	3	3
		9
73	9	3

(*d.*) Example X. — Find the price of 567 articles, at £2 6*s.* 8*d.* each.

Price at £1 each=£567; at £2 each=£567×2=£1,134; at 6*s.* 8*d.* (⅓ of £1) each=⅓ of £567=£189; and at £2 6*s.* 8*d.* (£2+6*s.* 8*d.*) each=£1,134+£189=£1,323:

567 @ £2 6*s.* 8*d.* each.

	£	
6*s.* 8*d.*=⅓ of £1	567	= price @ £1 each.
	2	
	1134	= „ „ £2 „
	189	= „ „ 6*s.* 8*d.* „
	1323	= „ „ £2 6*s.* 8*d.* each.

(*d*.) EXAMPLE XI.—Find the price of 93 articles, at £1 15*s*. each.

The excess (5*s*.) of £2 over £1 15*s*. being an aliquot part of £1, we regard the given price as the difference between £2 and 5*s*. Price at £1 each=£93; at £2 each=£93×2=£186; at 5*s*. (¼ of £1) each=¼ of £93=£23 5*s*.; and at £1 15*s*. (£2 − 5*s*.) each=£186 − £23 5*s*.=£162 15*s*.:

93 @ £1 15*s*. each.

	£					
5*s*.=¼ of £1	93	= price	@	£1	each.
	2				
	186	=	,,	,,	£2	,,
	23 5*s*.	=	,,	,,	5*s*.	,,
	162 15*s*.	=	,,	,,	£1 5*s*.	each.

(*d*.) EXAMPLE XII.—Find the price of 465 articles, at £3 17*s*. 4½*d*. each.

Price at £1 each=£465; at £3 each=£465×3=£1395; at 10*s*. (½ of £1) each=½ of £465=£232 10*s*.; at 5*s*. (½ of 10*s*.) each=½ of £232 10*s*.=£116 5*s*.; at 2*s*. (⅕ of 10*s*.) each =⅕ of £232 10*s*.=£46 10*s*.: at 4*d*. (⅙ of 2*s*.) each=⅙ of £46 10*s*.=£7 15*s*.; at ½*d*. (⅛ of 4*d*.) each=⅛ of £7 15*s*.= 19*s*. 4½*d*.; and at £3 17*s*. 4½*d*. (£3+10*s*.+5*s*.+2*s*.+4*d*.+½*d*.) each=£1395+£232 10*s*.+£116 5*s*.+£46 10*s*.+£7 15*s*.+ 19*s*. 4½*d*.=£1798 19*s*. 4½*d*.:

465 @ £3 17*s*. 4½*d*. each.

	£	*s*.	*d*.					
10*s*.=½ of £1	465	0	0	=	price	@	£1	each.
			3					
	1395	0	0	=	,,	,,	£3	,,
5*s*.=½ ,, 10*s*.	232	10	0	=	,,	,,	10*s*.	,,
2*s*.=⅕ ,, 10*s*.	116	5	0	=	,,	,,	5*s*.	,,
4*d*.=⅙ ,, 2*s*.	46	10	0	=	,,	,,	2*s*.	,,
½*d*.=⅛ ,, 4*d*.	7	15	0	=	,,	,,	4*d*.	,,
	0	19	4½	=	,,	,,	½*d*.	,,
	1798	19	4½	=	,,		£3 17*s*. 4½*d*.	each.

Class 2, in which the quantity whose price is required is expressed by a simple number of a higher or lower denomina-

tion than (but of the *same kind* as) that of the unit whose price is given.

EXAMPLE XIII.—What is the price of 9 *cwt.* of beef, at $7\frac{1}{2}d.$ a *pound?*

The 9 cwt. being reduced to ($112 \times 9 =$) 1008 lbs., this question becomes an exercise in Class 1:

1008 lbs. @ $7\frac{1}{2}d.$ a pound.

	s.	
$6d. = \frac{1}{2}$ of $1s.$	1008	= price @ $1s.$ a pound.
$1\frac{1}{2}d. = \frac{1}{4}$ „ $6d.$	504	= „ „ $6d.$ „
	126	= „ „ $1\frac{1}{2}d.$ „
	630 or £31 10	= „ „ $7\frac{1}{2}d.$ „

EXAMPLE XIV.—What is the price of 13 *dwts.* of silver, at $5s.\ 8d.$ an *ounce?*

Breaking up 13 dwts. into 10 dwts. ($\frac{1}{2}$ of 1 oz.), 2 dwts. ($\frac{1}{5}$ of 10 dwts.), and 1 dwt. ($\frac{1}{2}$ of 2 dwts.), we find the price of 10 dwts. $= \frac{1}{2}$ of $5s.\ 8d. = 2s.\ 10d.$; the price of 2 dwts. $= \frac{1}{5}$ of $2s.\ 10d. = 6{\cdot}8d.$; the price of 1 dwt. $= \frac{1}{2}$ of $6{\cdot}8d. = 3{\cdot}4d.$; and the price of ($10+2+1=$) 13 dwts. $= 2s.\ 10d. + 6{\cdot}8d. + 3{\cdot}4d. = 3s.\ 8d.$:

13 dwts. @ $5s.\ 8d.$ an ounce.

dwts.	*s.*	*d.*	
$10 = \frac{1}{2}$ of 1 oz.	5	8	= price of 1 oz.
$2 = \frac{1}{5}$ „ 10 dwts.	2	10	= „ „ 10 dwts.
$1 = \frac{1}{2}$ „ 2 „	0	6·8	= „ „ 2 „
	0	3·4	= „ „ 1 „
	3	8·2	= „ „ 13 „

Class 3, in which the quantity whose price is required is expressed by a fractional, a mixed, or a compound number.

EXAMPLE XV.—Find the price of $19\frac{7}{8}$ yds. of cloth, at $11s.\ 2d.$ a yard.

We first find the price of 19 yds., and then add the price of $\frac{7}{8}$ of a yard. Price of 19 yds. at £1 a yard $=$ £19; at $10s.$ ($\frac{1}{2}$ of £1) a yard $= \frac{1}{2}$ of £19 $=$ £9 $10s.$; at $1s.$ ($\frac{1}{10}$ of $10s.$) a yard $= \frac{1}{10}$ of £9 $10s. = 19s.$; at $2d.$ ($\frac{1}{6}$ of $1s.$) a yard $= \frac{1}{6}$ of $19s. = 3s.\ 2d.$; and at ($10s. + 1s. + 2d. =$) $11s.\ 2d.$ a yard $=$ £9 $10s.$

$+19s.+3s.\ 2d.=£10\ 12s.\ 2d.$ Price of $\frac{7}{8}$ of a yard $=\frac{7}{8}$ of $11s.\ 2d.$ $=11s.\ 2d.\ \times 7 \div 8 = 78s.\ 2d. \div 8 = 9s.\ 9\frac{1}{4}d.$ Price of $(19+\frac{7}{8}=)$ $19\frac{7}{8}$ yds. at $11s.\ 2d.$ a yard $=£10\ 12s.\ 2d.+9s.\ 9\frac{1}{4}d.=£11\ 1s.\ 11\frac{1}{4}d.$:

$19\frac{7}{8}$ yds. @ $11s.\ 2d.$ a yard.

	£	s.	d.	
$10s.=\frac{1}{2}$ of £1	19	0	0	= price of 19 yds. @ £1 a yard.
$1s.=\frac{1}{10}$ „ 10s.	9	10	0	= „ „ 10s. „
$2d.=\frac{1}{6}$ „ 1s.	0	19	0	= „ „ 1s. „
	0	3	2	= „ „ 2d. „
	10	12	2	= price, @ 11s. 2d. a yard, of 19 yds.
$\frac{7}{8}$ of 11s. 2d. =	0	9	$9\frac{1}{4}$	= price, @ 11s. 2d. a yard, of $\frac{7}{8}$ „
	11	1	$11\frac{1}{4}$	= price, @ 11s. 2d. a yard, of $19\frac{7}{8}$ „

This result is more easily obtained when—regarding $19\frac{7}{8}$ yds. as the difference between 20 yds. and $\frac{1}{8}$ of a yard—we subtract the price of $\frac{1}{8}$ of a yard from the price (found by means of Compound Multiplication) of 20 yds.:

	£	s.	d.	
	0	11	2	= price of 1 yd.
			4	
	2	4	8	
			5	
	11	3	4	= „ „ 20 yds
$\frac{1}{8}$ of 11s. 2d. =	0	1	$4\frac{3}{4}$	= „ „ $\frac{1}{8}$ yd.
	11	1	$11\frac{1}{4}$	= „ „ $19\frac{7}{8}$ yds.

Example XVI.—Find the rent of 35A. 3R. 30P. of land, at £2 17s. 6d. an acre.

If the rent of an acre were £1, the rent of a rood ($\frac{1}{4}$ of an acre) would be $\frac{1}{4}$ of £1 = 5s., whilst the rent of a perch ($\frac{1}{40}$ of a rood) would be $\frac{1}{40}$ of $5s.=1\frac{1}{2}d.$ So that when we set down £1 for every acre, 5s. for every rood, and $1\frac{1}{2}d.$ for every perch, we have, in the sum of the three amounts, the rent of the given quantity of land at £1 an acre: $£35+3\times 5s.+30\times 1\frac{1}{2}d.=£35\ 18s.\ 9d.$ At £2 an acre, therefore, the rent would be $£35\ 18s.\ 9d.\times 2=£71\ 17s.\ 6d.$; at 10s. ($\frac{1}{2}$ of £1) an acre, $\frac{1}{2}$ of $£35\ 18s.\ 9d.=£17\ 19s.\ 4\frac{1}{2}d.$; at 5s. ($\frac{1}{2}$ of 10s.) an acre, $\frac{1}{2}$ of $£17\ 19s.\ 4\frac{1}{2}=£8\ 19\ 8\frac{1}{4}d.$; at 2s. 6d. ($\frac{1}{2}$ of 5s.) an acre=

½ of £8 19*s*. 8¼*d*.=£4 9*s*. 10⅛; and at (£2+10*s*.+5*s*.+2*s*. 6*d*. =) £2 17*s*. 6*d*. an acre, £71 17*s*. 6*d*.+£17 19*s*. 4½*d*.+£8 19*s*. 8¼*d*.+£4 9*s*. 10⅛*d*.=£103 6*s*. 4⅞*d*.:

A.	R.	P.		
35	3	30	@	£2 17*s*. 6*d*. an acre.
£1	5*s*.	1½*d*.		

[£35+15*s*.+3*s*. 9*d*.=]

	£	*s*.	*d*.		
10*s*. = ½ of £1	35	18	9	= rent @	£1 an acre.
			2		
	71	17	6 =	,,	£2 ,,
5*s*. = ½ ,, 10*s*.	17	19	4½=	,,	10*s*. ,,
2*s*. 6*d*. = ½ ,, 5*s*.	8	19	8¼=	,,	5*s*. ,,
	4	9	10⅛=	,,	2*s*. 6*d*. ,,
	103	6	4⅞=	,,	£2 17*s*. 6*d*. an acre.

This result is more easily obtained when—regarding £2 17*s*. 6*d*. as the difference between £3 and 2*s*. 6*d*., which is an aliquot part of £1—we subtract the rent at 2*s*. 6*d*. an acre from the rent at £3 an acre. Multiplying £35 18*s*. 9*d*., the rent at £1 an acre, by 3, we find the rent at £3 an acre to be £107 16*s*. 3*d*.; dividing £35 18*s*. 9*d*. by 8, we find the rent at 2*s*. 6*d*. (⅛ of £1) an acre to be £4 9*s*. 10⅛*d*.; and, subtracting £4 9*s*. 10⅛*d*. from £107 16*s*. 3*d*., we find the rent at (£3—2*s*. 6*d*.=) £2 17*s*. 6*d*. an acre to be £107 16*s*. 3*d*.—£4 9*s*. 10⅛*d*.=£103 6*s*. 4⅞*d*.:

	£	*s*.	*d*.		
2*s*. 6*d*.=⅛ of £1	35	18	9	= rent @	£1 an acre.
			3		
	107	16	3 =	,,	£3 ,,
	4	9	10⅛ =	,,	2*s*. 6*d*. ,,
	103	6	4⅞ =	,,	£2 17*s*. 6*d*. ,,

The following is a third solution, which, however, involves the employment of a larger divisor than 12. Regarding the given quantity of land as the difference between 36 acres and 10 perches—an aliquot part of an acre, we find the answer by subtracting the rent of 10 perches from the rent of 36 acres. Rent of 36 acres=£2 17*s*. 6*d*. × 36=£2 17*s*. 6*d*. × 12 × 3= £103 10*s*.; rent of 10 perches ($\frac{1}{16}$ of an acre)=$\frac{1}{16}$ of £2 17*s*. 6*d*. =3*s*. 7⅛*d*.; rent of 35A. 3R. 30P.=£103 6*s*. 4⅞*d*.:

	£	s.	d.		
10P. = $\frac{1}{16}$ of 1A.	2	17	6	=	rent of 1 acre.
			12		
	34	10	0		
			3		
	103	10	0	=	„ „ 36 acres.
	0	3	$7\frac{1}{8}$	=	„ „ 10 perches.
	103	6	$4\frac{7}{8}$	=	„ „ 35A. 3R. 30P.

EXAMPLE XVII.—Find the price of 29 cwt. 2 qrs. 18 lbs. of sugar, at £2 7*s.* 10*d.* a cwt.

If the price of 1 cwt. were £1, the price of 1 qr. ($\frac{1}{4}$ of 1 cwt.) would be $\frac{1}{4}$ of £1=5*s.*, and the price of 1 lb. ($\frac{1}{28}$ of 1 qr.) $\frac{1}{28}$ of 5*s.*=$2\frac{1}{7}$*d.* So that when we set down £1 for each hundred-weight, 5*s.* for each quarter, and $2\frac{1}{7}$*d.* for each pound, we have, in the sum of the three amounts, the price of the given quantity of sugar at £1 a cwt.: £29+2×5*s.*+18×$2\frac{1}{7}$*d.* =£29+10*s.*+38·57*d.*=£29 13*s.* 2·57*d.* At £2 a cwt., the price would be £29 13*s.* 2·57*d.*×2=£59 6*s.* 5·14*d.*; at 5*s.* ($\frac{1}{4}$ of £1) a cwt., $\frac{1}{4}$ of £29 13*s.* 2·57*d.*=£7 8*s.* 3·64*d.*; at 2*s.* ($\frac{1}{10}$ of £1) a cwt., $\frac{1}{10}$ of £29 13*s.* 2·57*d.*=£2 19*s.* 3·86*d.*;* at 8*d.* ($\frac{1}{3}$ of 2*s.*) a cwt., $\frac{1}{3}$ of £2 19*s.* 3·86*d.*=19*s.* 9·28*d.*; and at 2*d.* ($\frac{1}{4}$ of 8*d.*) a cwt., $\frac{1}{4}$ of 19*s.* 9·28*d.*=4*s.* 11·32*d.* Therefore, at (£2+5*s.*+2*s.*+8*d.*+2*d.*)=£2 7*s.* 10*d.* a cwt., the price must be £59 6*s.* 5·14*d.*+£7 8*s.* 3·64*d.*+£2 19*s.* 3·86*d.*+19*s.* 9·28*d.*+4*s.* 11·32*d.* =£70 18*s.* 9·24*d.*, or (very nearly) £70 18*s.* $9\frac{1}{4}$*d.*:

cwt.	qrs.	lbs.	
29	2	18	@ £2 7*s.* 10*d.*
£1	5*s.*	$2\frac{1}{7}$*d.*	

[£29+10*s.*+38·57*d.*=]

	£	s.	d.				
5*s.*=$\frac{1}{4}$, 2*s.*=$\frac{1}{10}$ of £1	29	13	2·57	=	price @ £1 a cwt		
			2				
	59	6	5·14	=	„	£2	„
	7	8	3·64	=	„	5*s.*	„
8*d.*=$\frac{1}{3}$ of 2*s.*	2	19	3·86*	=	„	2*s.*	„
2*d.*=$\frac{1}{4}$ „ 8*d.*	0	19	9·28	=	„	8*d.*	„
	0	4	11·32	=	„	2*d.*	„
	70	18	9·24	=	„	£2 7*s.* 10*d.*	
	or 70	18	$9\frac{1}{4}$			a cwt.	

* The figure 6, although too high, is nearer to the truth than 5.

NOTE 1.—When extreme accuracy is required, smaller amounts than 1*d.* should each, as in the last example, be expressed as a decimal (carried not farther than two places) of 1*d.*; and the resulting decimal in the answer, when too large to be rejected, should afterwards be converted into $\frac{1}{4}$*d.*, $\frac{1}{2}$*d.*, $\frac{3}{4}$*d.*, or 1*d.**—according to the value of the decimal: ·25*d.*=$\frac{1}{4}$*d.*; ·5*d.*=$\frac{1}{2}$*d.*; ·75*d.*=$\frac{3}{4}$*d.*

NOTE 2.—Notwithstanding what has just been said, the pupil, in working examples like the last, should occasionally be required—if only as an exercise in Fractions—to express smaller sums than 1*d.* as fractions of 1*d.* Thus, returning to the last example, and writing $18 \times 2\frac{1}{7}$*d.* as ($38\frac{4}{7}$*d.*=) 3*s.* $2\frac{4}{7}$*d.*, we have—

	£	s.	d.		
5*s.*=$\frac{1}{4}$ } of £1 2*s.*=$\frac{1}{10}$ }	29	13	$2\frac{4}{7}$	=	price @ £1 a cwt.
			2		
	59	6	$5\frac{1}{7}$	=	,, £2 ,,
	7	8	$3\frac{9}{14}$ (*a*)	=	,, 5*s.* ,,
8*d.*=$\frac{1}{3}$ of 2*s.*	2	19	$3\frac{6}{7}$ (*b*)	=	,, 2*s.* ,,
2*d.*=$\frac{1}{4}$,, 8*d.*	0	19	$9\frac{2}{7}$ (*c*)	=	,, 8*d.* ,,
	0	4	$11\frac{9}{28}$ (*d*)	=	,, 2*d.* ,,
	70	18	$9\frac{7}{28}$ }	=	,, { £2 7*s.* 10*d.*
	or 70	18	$9\frac{1}{4}$ }		a cwt.

The first two fractions ($\frac{4}{7}$ and $\frac{1}{7}$) explain themselves. The others require very little explanation:—

(*a*) After setting down £7 8*s.* 3*d.* as $\frac{1}{4}$ of £29 13*s.* $2\frac{4}{7}$*d.*, we have $2\frac{4}{7}$*d.* remaining. The division of this remainder by 4 gives $\frac{9}{14}$*d.* for quotient: $2\frac{4}{7}$*d.*$\div 4 = \frac{18}{7}$*d.*$\div 4 = \frac{18}{28}$*d.*$= \frac{9}{14}$*d.*

(*b*) After setting down £2 19*s.* 3*d.* as $\frac{1}{10}$ of £29 13*s.* $2\frac{4}{7}$*d.*, we have $8\frac{4}{7}$*d.* remaining. The division of this remainder by 10 gives $\frac{6}{7}$*d.* for quotient: $8\frac{4}{7}$*d.*$\div 10 = \frac{60}{7}$*d.*$\div 10 = \frac{60}{70}$*d.*$= \frac{6}{7}$*d.*

(*c*) After writing 19*s.* 9*d.* as $\frac{1}{3}$ of £2 19*s.* $3\frac{6}{7}$*d.*, we have $\frac{6}{7}$*d.* remaining. The division of this remainder by 3 gives $\frac{2}{7}$*d.* for quotient: $\frac{6}{7}$*d.*$\div 3 = \frac{6}{21}$*d.*$= \frac{2}{7}$*d.*

(*d*) After writing 4*s.* 11*d.* as $\frac{1}{4}$ of 19*s.* $9\frac{2}{7}$*d.*, we have $1\frac{2}{7}$*d.* remaining. The division of this remainder by 4 gives $\frac{9}{28}$*d.* for quotient: $1\frac{2}{7}$*d.*$\div 4 = \frac{9}{7}$*d.*$\div 4 = \frac{9}{28}$*d.*

The addition of the fractions $\frac{1}{7}$, $\frac{9}{14}$, $\frac{6}{7}$, $\frac{2}{7}$, and $\frac{9}{28}$ presents no difficulty: $\frac{1}{7}+\frac{9}{14}+\frac{6}{7}+\frac{2}{7}+\frac{9}{28}=\frac{4}{28}+\frac{18}{28}+\frac{24}{28}+\frac{8}{28}+\frac{9}{28}=\frac{63}{28}=2\frac{7}{28}$, or $2\frac{1}{4}$.

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* When the figure in the first decimal place is 9, the amount is nearer to 1*d.* than to $\frac{3}{4}$*d.*

INTEREST.

175. The charge made for a loan of money is called INTEREST.

176. The money lent is termed the PRINCIPAL; the sum of the Principal and its Interest for a given time is called the AMOUNT for that time; and the interest of £100 for a year is known as the RATE PER CENT. *per annum*—the words "per annum," however, being usually omitted.

177. Interest is either SIMPLE or COMPOUND. When charged upon the Principal only, and paid in yearly, half-yearly,* or quarterly instalments, (according to agreement,) Interest is called *Simple;* but when such instalments, on becoming due, are added to the Principal, and made to bear interest at the same rate, money is said to be at *Compound* Interest.

SIMPLE INTEREST.

EXAMPLE I.—What interest would £78 9*s*. 10*d*. produce in a year, at $3\frac{1}{2}$ per cent. (per annum)?

At the given rate, £100 would produce £$3\frac{1}{2}$ in a year: at the same rate, therefore, and in the same time, £78 9*s*. 10*d*. would produce a sum as many times less than £$3\frac{1}{2}$ as £78 9*s*. 10*d*. is less than £100. The answer being represented by a, we thus have the proportion—

$$£100 : £78\ 9s.\ 10d. :: £3\tfrac{1}{2} : a,$$

or, by Alternation (see p. 169),

$$£100 : £3\tfrac{1}{2} :: £78\ 9s.\ 10d. : a$$

From this last proportion† we find $a = £78\ 9s.\ 10d. \times 3\frac{1}{2} \div 100 = £2\ 14s.\ 11\frac{1}{4}d.$

Or thus: £78 9*s*. 10*d*.=£78·492 (§ 87); £$3\frac{1}{2}$=£3·5; a= £78·492×3·5÷100=£2·747$\frac{2}{9}\frac{2}{5}$=£2 14*s*. $11\frac{1}{4}$*d*. (§ 88).

Here the Principal is £78 9*s*. 10*d*.; the Interest, £2 14*s*. $11\frac{1}{4}$*d*.; the Amount, (£78 9*s*. 10*d*.+£2 14*s*. $11\frac{1}{4}$*d*.=) £81 4*s*. $9\frac{1}{4}$*d*.; and the Rate per cent., £$3\frac{1}{2}$, or £3 10*s*.

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* As a general rule, (Simple) Interest—like rent—is paid *half-yearly*.

† The first proportion would involve the reduction of the first two terms (£100 and £78 9*s*. 10*d*.) to pence.

178. To find the Interest of any Principal for a Year, at any Rate per cent.: Multiply the Principal by the Rate per cent., and divide the product by 100.

NOTE 1.—To find the interest for two or more years, we *multiply the interest for one year by the number of years.* Thus, at $3\frac{1}{2}$ per cent., £78 9*s.* 10*d.* would produce—in 2 years, £2 14*s.* $11\frac{1}{4}d. \times 2 =$ £5 9*s.* $10\frac{1}{2}d.$; in 3 years, £2 14*s.* $11\frac{1}{4}d. \times 3 =$ £8 4*s.* $9\frac{3}{4}d.$; &c.

NOTE 2.—When the rate is 5 per cent., a year's interest can be found with great facility. Instead of multiplying the principal by 5, and dividing the product by 100, we simply take $\frac{1}{20}$ of the principal; and this we do by setting down 1*s.* for every pound, 3*d.* for every crown, and $\frac{1}{4}d.$ for every five-pence in the principal. Thus, in £345 12*s.* 7*d.* there are 345 pounds, 2 crowns, and (2*s.* 7*d.*=31*d.*=) 6 five-pences. As the interest of this sum for a year, therefore, at 5 per cent., we write 345 shillings+2 three-pences+6 farthings=£17 5*s.*+6*d.*+$1\frac{1}{2}d.$= £17 5*s.* $7\frac{1}{2}d.$

From 5 per cent. we can easily pass to either 4 or 6 per cent., by subtracting or adding the interest at 1 per cent.—obtained from the division of the interest at 5 per cent. by 5. Thus, interest of £345 12*s.* 7*d.* (for a year) at 5 per cent.=£17 5*s.* $7\frac{1}{2}d.$; at 1 per cent.=£17 5*s.* $7\frac{1}{2}d. \div 5 =$ £3 9*s.* $1\frac{1}{2}d.$; at 4 per cent.= £17 5*s.* $7\frac{1}{2}d.$ − £3 9*s.* $1\frac{1}{2}d.$=£13 16*s.* 6*d.*; and at 6 per cent.= £17 5*s.* $7\frac{1}{2}d.$+£3 9*s.* $1\frac{1}{2}d.$=£20 14*s.* 9*d.*

NOTE 3.—The time, when not an exact number of years, is always expressed in DAYS, or in years *and* DAYS. Notwithstanding what several treatises on Arithmetic imply to the contrary, there is no such thing in the commercial world as a *month's*—or a number of *months'*—interest.* Take, for instance, three "months"—(*a*) from the 1st of February to the 1st of March; (*b*) from the 1st of March to the 1st of April; and (*c*) from the 1st of April to the 1st of May. In the calculation of interest, the first of these periods would be expressed as 28 days, the second as 31 days, and the third as 30 days. Because money borrowed on the 1st of February, and repaid on the 1st of March, would have been at interest during the last 27 days of February and the first day of March; money borrowed on the 1st of March, and repaid on the 1st of April, would have been at in-

* This appears to be contradicted by the fact that bills of exchange are usually "drawn" for a number of months; but when a bill comes to be discounted, the discount—as we shall find hereafter—is invariably calculated for days. (See p. 202.)

terest during the last 30 days of March and the first day of April; and money borrowed on the 1st of April, and repaid on the 1st of May, would have been at interest during the last 29 days of April and the first day of May:—

	dys.		dys.		dys.
Feb., ...	27	Mar., ...	30	Apl., ...	29
Mar., ...	1	Apl., ...	1	May, ...	1
	28		31		30

The *first* day—that on which the borrower received the principal—is never included in the number of days for which interest is charged; the reason being that money lent on (say) the 1st of February would not have been at interest for a day until the 2nd of February.

EXAMPLE II.—What interest ought to be paid for the use of £678 from the 17th of April till the 23rd of July, at 4 per cent. (per annum)?

Having found the "time" to be 97 days, we can put the question in this way: The interest of £100 for 365 days being £4, what is the interest of £678 for 97 days? This is merely an exercise in compound proportion, by means of which—as shown below—we find the answer to be $£\frac{678 \times 97 \times 4}{36500}$, or (the terms of the fraction being both multiplied by 2) $£\frac{678 \times 97 \times 8}{73000}$.

	dys.
April, ...	13
May, ...	31
June, ...	30
July, ...	23
	97

It will be seen that the numerator of this last fraction is the continued product of the principal (£678), the number of days (97), and twice the rate ($8=4\times2$):—

£	dys.	£
100	365	4
678	97	?

$$\left.\begin{array}{l}100 : 678 \\ 365 : \ \ 97\end{array}\right\} :: 4 : a$$

$$36500 : 678\times97 :: 4 : a$$

$$a = £\frac{678\times97\times4}{36500} = £\frac{678\times97\times4\times2}{36500\times2} = £\frac{678\times97\times8}{73000}.$$

179. To find the Interest of a given Principal for a given number of Days, at any Rate per cent.: Multiply the Principal by the number of Days, and the product by twice the Rate; then, divide the result by 73,000.

NOTE.—By doubling the terms of the fraction obtained from the compound proportion, we have 73,000 for divisor, instead of 36,500, which would be found less convenient; and it is quite as easy to multiply by twice the rate as by the rate itself—indeed, easier in many cases. In dividing by 73,000, a schoolboy would naturally proceed as follows—employing as factors 1,000 and 73:

$$678\times97\times8=526128;\ 526128\div1{,}000=526{\cdot}128;$$
$$526{\cdot}128\div73=7{\cdot}207;\ £7{\cdot}207=£7\ 4s.\ 1\tfrac{3}{4}d.$$

Here, however, is a less troublesome way of dividing 526128 by 73,000: Taking a third of 526128, we obtain 175376; taking a tenth of 175376,* we obtain 17537; and, taking a tenth of 17537,* we obtain 1753. Then, adding the four numbers together, and dividing their sum by 100,000, we find the answer to be, as before, £7·207, or £7 4s. 1¾d.

$$\begin{array}{r} 526128 \\ 175376 \\ 17537 \\ 1753 \\ \hline 7{\cdot}20794 \end{array}$$

The principle involved in this is easily understood. We merely, before performing the division, multiply both the dividend and the divisor by $1+\frac{1}{3}+\frac{1}{30}+\frac{1}{300}$:

$$\begin{array}{lr}
526128\times1= & 526128 \\
526128\times\frac{1}{3}=\frac{1}{3}\text{ of }526128= & 175376 \\
\left.\begin{array}{r}526128\times\frac{1}{30}=\frac{1}{30}\text{ of }526128=\frac{1}{10}\text{ of }\frac{1}{3}\text{ of }526128= \\ \frac{1}{10}\text{ of }175376=\end{array}\right\} & 17537 \\
\left.\begin{array}{r}526128\times\frac{1}{300}=\frac{1}{300}\text{ of }526128=\frac{1}{10}\text{ of }\frac{1}{30}\text{ of }526128= \\ \frac{1}{10}\text{ of }17537=\end{array}\right\} & 1753 \\
\hline
526128\times\left(1+\frac{1}{3}+\frac{1}{30}+\frac{1}{300}\right)= & 720794
\end{array}$$

$$\begin{array}{lr}
73000\times1= & 73000 \\
73000\times\frac{1}{3}=\frac{1}{3}\text{ of }73000= & 24333\frac{1}{3} \\
\left.\begin{array}{r}73000\times\frac{1}{30}=\frac{1}{30}\text{ of }73000=\frac{1}{10}\text{ of }\frac{1}{3}\text{ of }73000= \\ \frac{1}{10}\text{ of }24333\frac{1}{3}=\end{array}\right\} & 2433\frac{1}{3} \\
\left.\begin{array}{r}73000\times\frac{1}{300}=\frac{1}{300}\text{ of }73000=\frac{1}{10}\text{ of }\frac{1}{30}\text{ of }73000= \\ \frac{1}{10}\text{ of }2433\frac{1}{3}=\end{array}\right\} & 243\frac{1}{3} \\
\hline
73000\times\left(1+\frac{1}{3}+\frac{1}{30}+\frac{1}{300}\right)= & 100010\dagger
\end{array}$$

Money lent on a Deposit Receipt.—Instead of being lent for a definite period, and at a fixed rate of interest, money is often given to a banker upon the understanding that it can be with-

* By simply removing each figure a place to the right.

† This number being larger by 10 than 100,000, the quotient obtained when we divide by 100,000 is slightly in excess of the truth. For practical purposes, however, the error is so small as to be unworthy of notice. The last example is an illustration of this: £7·20794 exceeds £7·207 by less than a farthing.

drawn at any time, and that, if not withdrawn within a month, it shall bear interest at a variable rate—to be determined, from day to day, at the banker's discretion, by the state of the money-market. Money employed in this way is said to be lent on a "deposit receipt"—the name of the acknowledgment given to the lender by the banker.

Example III.—A sum of £234 was lent on a deposit receipt, and the rate was 2 per cent. for the first 11 days, $1\frac{1}{2}$ per cent. for the next 17 days, $2\frac{1}{2}$ per cent. for the next 8 days, 3 per cent. for the next 25 days, 4 per cent. for the next 9 days, and $3\frac{1}{2}$ per cent. for the remaining 13 days; find the interest.

Here we have no fewer than six different rates—each involving, apparently, as much work as was necessary in the last example. The answer can be found, however, in the way shown in the margin. Multiplying each number of days by twice the corresponding rate, and adding the products together, we obtain 448, which, when the last figure (8) is "cut off," becomes 44. We then set down, as the required interest, what the given principal would produce in 44 days, at 5 per cent.*

per cent.		dys.	
2	for	11	$2 \times 2=4;\ 4\times 11=44$
$1\frac{1}{2}$	,,	17	$1\frac{1}{2}\times 2=3;\ 3\times 17=51$
$2\frac{1}{2}$	,,	8	$2\frac{1}{2}\times 2=5;\ 5\times 8=40$
3	,,	25	$3 \times 2=6;\ 6\times 25=150$
4	,,	9	$4 \times 2=8;\ 8\times 9=72$
$3\frac{1}{2}$	,,	13	$3\frac{1}{2}\times 2=7;\ 7\times 13=91$
			448

Interest of £234 for 44 days, at 5 per cent. $= \frac{£234\times 44\times 10}{73000} =$ £1 8*s.* $2\frac{1}{2}$*d.*, the answer.

Let us now consider the reason of this:

In how many days, at 5 per cent., would a principal produce as much interest as would be produced by the same principal in 11 days, at 2 per cent.?

$$5 : 2 :: 11 : a;\ a=\frac{11\times 2}{5}=\frac{11\times 4}{10}.$$

In how many days, at 5 per cent., would a principal produce as much interest as would be produced by the same principal in 17 days, at $1\frac{1}{2}$ per cent.?

$$5 : 1\tfrac{1}{2} :: 17 : a;\ a=\frac{17\times 1\frac{1}{2}}{5}=\frac{17\times 3}{10}.$$

In how many days, at 5 per cent., would a principal produce as much interest as would be produced by the same principal in 8 days, at $2\frac{1}{2}$ per cent.?

$$5 : 2\tfrac{1}{2} :: 8 : a;\ a=\frac{8\times 2\frac{1}{2}}{5}=\frac{8\times 5}{10}.$$

* A bank clerk, instead of calculating this, would take it from an Interest table.

In the same way it could be shown that 25 days, when the rate is 3 per cent., are equivalent to $\frac{25\times 6}{10}$ days when the rate is 5 per cent.; that 9 days, when the rate is 4 per cent., are equivalent to $\frac{9\times 8}{10}$ days when the rate is 5 per cent.; and that 13 days, when the rate is $3\frac{1}{2}$ per cent., are equivalent to $\frac{13\times 7}{10}$ days when the rate is 5 per cent. So that 2 per cent. for 11 days, $1\frac{1}{2}$ per cent. for 17 days, $2\frac{1}{2}$ per cent. for 8 days, 3 per cent. for 25 days, 4 per cent. for 9 days, and $3\frac{1}{2}$ per cent. for 13 days—all taken together—are equivalent to 5 per cent. for $\frac{1}{10}$ of $(44+51+40+150+72+91=)$ 448 days; and we take $\frac{1}{10}$ of 448 by cutting off the last figure.

Principal, Interest, Rate, Time: Any one of these can be found, when the remaining three are given. For, putting P for the Principal, I for the Interest, R for the Rate, and D for the Time—expressed in *days*, we have (§179) $\frac{P\times D\times 2R}{73{,}000}=I$; $[P\times D\times 2R=73{,}000\times I;]$ $P=\frac{73{,}000\times I}{D\times 2R}$; $R=\frac{73{,}000\times I}{P\times D\times 2}$; and $D=\frac{73{,}000\times I}{P\times 2R}$.

180. To find the Principal, when the Interest, Rate, and Time are given: Set down, for dividend, the product of the Interest by 73,000; and, for divisor, the product of the number of Days by twice the Rate. The resulting quotient will be the answer.

Example IV.—Find the principal which, at 3 per cent., would produce 19*s.* 7*d.* in 124 days.

Writing 19*s.* 7*d.* as £·979, and multiplying by 73,000, we obtain, for dividend, $(73{,}000\times{\cdot}979=)$ 71467. For divisor, we take the product of 124 by the double of 3—that is, $(124\times 6=)$ 744. Then, dividing 71467 by 744, we find the required principal to be $(71467\div 744=)$ £96 1*s.* 2*d.*, nearly.

181. To find the Rate, when the Principal, Interest, and Time are given: Set down, for dividend, the product of the Interest by 73,000; and, for divisor, twice the product of the Principal by the number of Days. The resulting quotient will be the answer.

Example V.—At what rate would £125 produce £1 10*s.* in 219 days.

Writing £1 10*s.* as £1·5, we set down, for dividend, 73,000 × 1·5=109500. For divisor, we take twice the product of 125 by 219—that is, (125×219×2=) 54750. Then, dividing 109500 by 54750, we find the required rate to be (109500÷54750=) 2 per cent.

182. To find the Time, when the Principal, Interest, and Rate are given: Set down, for dividend, the product of the Interest by 73,000; and, for divisor, the product of the Principal by twice the Rate. The resulting quotient will be the answer in days.

Example VI.—In what time would £750 produce £40 11*s.* 11*d.*, at 4½ per cent.?

Converting £40 11*s.* 11*d.* into £40·596, we have, for dividend, 73,000×40·596=2963508. For divisor, we write twice the product of 750 by 4½—that is, (750×4½×2=) 6750. Then, dividing 2963508 by 6750, we find the required time to be (2963508÷6750=) 439 days, or 1 year and 74 days.

Note.—Principal, Rate, Time: It will be observed that, in finding any one of these three, we simply divide twice the product of the remaining two into the product of the Interest by 73,000.

183. To determine the Principal, when the Amount, Rate, and Time are known: Say—as the amount of £100 for the stated time, and at the stated Rate, is to £100, so is the given Amount to the required Principal.

Example VII.—What principal would amount, in 7 years, at 5 per cent., to £1320 6*s.*?

At the given rate, £100 would produce £5 in one year, and (£5×7=) £35 in seven years; so that, for the given time, and at the given rate, the "amount" of £100 would be (£100+£35=) £135. The question now assumes this form: If £100, principal, would become £135, amount, under certain circumstances, what principal would become £1320 6*s.*, amount, under the *same* circumstances? The proportion is—

P.	A.
£100	£135
?	£1320 6*s.*

£135 : £1320 6*s.* : : £100 : *a*;

or, by Alternation,

£135 : £100 : : £1320 6*s.* : *a*

a=£1320·3×100÷135=£978.

If, instead of £100, we take any other sum—say £30, we shall obtain the same result, but at the expense of a little more trouble. At 5 per cent., £30 would produce 30*s*., or £1 10*s*. in one year, and (£1 10*s*. × 7=) £10 10*s*. in seven years; so that the amount of £30 for the given time, and at the given rate, would be (£30+£10 10*s*.=) £40 10*s*. We therefore say—as £40 10*s*., amount, is to £30, the principal which would produce it, so is £1320 6*s*., another amount, to the principal which would produce *it* under the same circumstances:

$$£40{\cdot}5 : £30 :: £1320{\cdot}3 : a$$
$$a = £1320{\cdot}3 \times 30 \div 40{\cdot}5 = £978, \text{ as before.}$$

DISCOUNT.

184. When a debt is paid before the time originally agreed upon, an abatement is usually made in the amount. This abatement is termed DISCOUNT; and the sum accepted in discharge of the debt is known as the *present-worth* of the debt. So that the full amount exceeds its present-worth by the discount.

EXAMPLE I.—A person holds a claim for £420, payable a year hence; what sum paid *now* would satisfy the claim, the rate of interest being 5 per cent.?

It is evident that the true present-worth of this claim is the sum which, if invested—as principal—at 5 per cent., would amount to £420 in a year. Now, at the given rate, £100 would amount to £105 in a year: consequently, the principal which, at the same rate, and in the same time, would amount to £420, is the fourth term of the proportion—

$$£105 : £100 :: £420 : a$$
$$a = (420 \times 100 \div 105 =) \ £400.$$

The *true* discount, therefore is £20—the difference between £420 and £400.

In the commercial world, however, discount is never calculated in this way; the amount of a debt being invariably treated (not as an "amount," which it really is, but) as a *principal*, and the discounter charging interest, not merely

upon what he advances as present-worth, but also upon a sum which he does *not* advance—that is, upon the discount itself. In the case under consideration, the sum charged as discount would be £21—the interest of £420 for a year, at 5 per cent.; so that the present-worth would be (£420—£21=) £399.*

185. Rule for the calculation of Discount: Find what interest the amount of the debt, if invested as principal, would produce in the given time and at the given rate; this interest will be the Discount required.

Bills of Exchange.—A bill of exchange may be described as a written engagement, by a debtor, in obedience to the order of a creditor, to pay a sum of money at a future time. The following is the usual form of a bill:—

[Stamp.]
£500 0 0

Sackville-street, Dublin,
21st April, 1870.

Three months after date, pay to me or my order the sum of five hundred pounds sterling—for value received.

John Jones.

James Browne.

To Mr. John Jones,
Patrick-street, Cork.

Jones would *accept* this bill by simply writing his name across the "face" (not the back) of it; and the money would be presumed to be payable at his house in Cork unless he—on the face of the bill—made a notification to the contrary, such as "Payable at the National Bank."

In relation to this bill, Browne (the creditor) and Jones (the debtor) would be spoken of as the DRAWER and the ACCEPTOR,† respectively.

When a bill is accepted, and returned to the drawer, what usually occurs is this: (*a*) The drawer "holds" the bill until it comes to "maturity"—that is, until the day on which payment

* It will be seen that £399 is the *true* present-worth—not of £420, but—of (£399+£19 19*s.*=) £418 19*s.*; the interest of £399 for a year, at 5 per cent., being £19 19*s.*

† The acceptor is sometimes called the *drawee.*

is to be demanded; (*b*) or he at once converts his claim into ready-money, by getting the bill discounted; (*c*) or he "passes" the bill, as an equivalent for cash, to a creditor of his own—this creditor, in his turn, being then at liberty to adopt, with respect to the bill, any one of the three courses just mentioned. Before, however, parting with a bill which has not arrived at maturity (in other words, before *negotiating* a bill), the drawer—as well as everybody else into whose hands it comes after leaving the drawer's—has to indorse it; that is, write his name on the back of it. So many as ten, fifteen, and even twenty names—representing an equal number of indorsers—are frequently met with on the back of a bill: a fact which enables us to understand the very important part which bills play in mercantile transactions.

In the United Kingdom, a bill "runs" three days—called DAYS OF GRACE—in addition to the time mentioned on the face of it. So that if a three months' bill were drawn on the 1st of July, the amount would not be legally due until the third day after the 1st of October—that is, until the 4th of October. If, however, the last of the days of grace happened to be a Sunday, the amount would be due on the preceding (Satur-) day. Moreover, if a four months' bill, say, were drawn on the 31st of May, the days of grace would be reckoned (not from the 31st of September—there being no such day, but) from the 30th of September; so that the amount would be due on the 3rd of October. Again, if a two months' bill were drawn on the 31st, 30th, or 29th of December, the two months would expire on the last day (28th or 29th, as the case may be) of February; so that the amount would be due on the 3rd of March.

EXAMPLE II.—A three months' bill for £240 was drawn on the 24th of June, and discounted on the 13th of July, at 6 per cent. per annum; find the discount.

The three months having expired on the 24th of September, the bill arrived at maturity three days later—that is, on the 27th of September; so that, when discounted (on the 13th of July), the bill had 76 days to run—namely, the last 18 days of July, the whole 31 days of August, and the first 27 days of September. The required

	dys.
July,	18
Aug.,	31
Sept.,	27
	76

£240 × 76 × 12 ÷ 73000 =
£218880 ÷ 73000 = £2·998 =
£2 19 11½:

$$\begin{array}{r} 218880 \\ 72960 \\ 7296 \\ 729 \\ \hline 2{\cdot}99865 \end{array}$$

discount, therefore, is the interest which £240 would produce in 76 days, at 6 per cent. per annum; and (§ 179) this we find to be £2 19 11½.

When a bill arrives at maturity, the holder—that is, the person in whose possession it happens to be at the time—applies to the acceptor, as a matter of course, for payment. In the event of his failing to obtain the amount, the holder at once takes the bill to a Notary Public,* by whom payment is again demanded on the same day, and who, if unsuccessful in his mission, *protests* the bill, by writing across the face of it—"Protested for non-payment." A written notification of the acceptor's default is next forwarded, without delay, to each of the indorsers, all of whom (the drawer included) are then bound—"jointly and severally"—to indemnify the holder. It is worthy of remark, however, that the indorsers would be released from all liability if the holder, in advising them of the non-acceptance of the bill, allowed any avoidable delay to occur—if, for instance, he wrote by a certain post, when he could have written by the preceding post. The indorsers would also be released from liability if the holder neglected to apply to the acceptor for payment on the *very day* on which the bill arrived at maturity. But such negligence on the holder's part would be no bar to subsequent proceedings against the acceptor.

When obliged to proceed against the indorsers for the recovery of the amount, (the acceptor having been found unable to pay,) the holder of a protested or "dishonoured" bill naturally selects, in the first instance, the person whom he considers most solvent. Should the full amount be obtained from this indorser, all the earlier indorsers would then be bound to indemnify him; but the later indorsers would be released from their liability.†

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* The services of a Notary Public can be—and occasionally are—dispensed with when the holder is himself prepared to prove that the bill was presented for payment at the proper time and place; but such proof is never required in the case of a bill which has passed through the hands of a Notary, whose noting is always accepted, in a court of law, as conclusive evidence on the point.

† Such bills as we have hitherto been considering are called INLAND bills, in contradistinction to another class, known as FOREIGN bills. By a "foreign" bill is meant—a bill drawn in one country, and accepted in another; the drawer, for instance, residing in London, and the acceptor in Paris; or the drawer residing in New York, and the acceptor in Dublin. A foreign bill is always made payable to a third person—called the PAYEE—residing in the same country as the acceptor. Thus, when (in the language of mercantile people) Smyth of New York draws *upon* Taylor of Dublin, *in favour of* Robinson—also of Dublin, Smyth is the "drawer;" Taylor, the "acceptor;" and Robinson, the "payee."

On account of the uncertainty and risk incidental to their transmis-

Promissory Notes.—A written promise to pay money on a future day is sometimes in the form of a promissory note, which runs thus :—

[Stamp.]

£500 0 0 *Patrick-street, Cork,*
21st *April,* 1870.

Three months after date, I promise to pay Mr. James Browne or order the sum of five hundred pounds sterling—for value received.

John Jones.

Jones (the debtor) would be called the MAKER of this promissory note, which would be quite as valuable to Browne as the bill of exchange referred to at page 201. In fact, the word "drawer" being omitted, and "maker" substituted for "acceptor," all that has been said about a bill of exchange is equally applicable to a promissory note—the two documents differing only in form. Latterly, however, promissory notes have been falling into disuse.

Every bank-note is a promissory note—*payable "on demand."* In the case of a promissory note which is payable on demand, there are no "days of grace."

STOCKS AND SHARES.

186. Formerly, when more money was required for Imperial purposes than could, at the time, be raised in taxes, the necessary sum was borrowed, at a certain rate of interest, upon the security of the State. All the outstanding loans thus obtained, from time to time, constitute what is called the NATIONAL DEBT, which now exceeds £800,000,000.

Notwithstanding the largeness of this amount,—occasioned chiefly by the wars with France and America,—there was no National Debt, properly so called, until the reign of William III. Previously to that time, loans for short periods used

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sion, foreign bills are usually drawn in *sets* of three each—the parts of a set being marked 1st, 2nd, 3rd, respectively, and forwarded by different ships. The part which first reaches its destination then becomes *the* bill, and is payable a certain number of days "after sight"—that is, after *presentation to,* and acceptance by, the acceptor. [See EXCHANGE.]

occasionally to be raised by the Sovereign, upon the security of the Crown revenues—the earliest such loan on record being one raised by Richard I. to defray the expenses of his crusade to the Holy Land; but no debt of a NATIONAL character had been contracted until the year 1694, when Parliament borrowed the capital of the Bank of England, £1,200,000,* at 8 per cent., on the understanding that, so long as the interest continued to be regularly paid, the State should be bound to no particular time for repayment of the principal.

The National Debt had increased to (in round numbers) £52,000,000 at the Peace of Utrecht, in 1713; to £79,000,000 at the Peace of Aix La Chapelle, in 1743; to £133,000,000 at the Peace of Paris, in 1763; to £249,000,000 at the close of the American War, in 1783; and to £770,000,000 at the close of the great war which lasted from 1793 till 1815. Since 1815, the largest debt contracted—at any one time—was a sum of £20,000,000, borrowed in 1835-6, and paid to the planters in the West Indies, as compensation for their losses consequent upon the abolition of slavery.

187. A person to whom the State owes any portion of the National Debt is said to be the holder of GOVERNMENT STOCK, or to have money in "*The Funds.*" Such a person receives interest at the rate of 5, 3½, 3, or 2½ per cent.—according to the kind of stock he holds.

The fact that there are different kinds of stock, bearing interest at different rates, is quite intelligible when we remember that the National Debt was not all contracted at the same time, and that the State was obliged—as private borrowers are obliged—to offer a higher rate of interest at one time than at another. Occasionally, too,—when the state of the money-market renders the scheme feasible,—a quantity of "old" is converted into "new" stock, and the rate of interest lowered; the holders of the old stock receiving due notice of the contemplated conversion, and the amount of his claim being paid, in cash, to anybody dissatisfied with the terms offered by Government.†

* The capital of the Bank has since increased very considerably—Government holding, at present, some £11,000,000 of it.

† Here is a parallel case: C lends D £1,000, at 5 per cent., on the understanding that the principal can be returned whenever D pleases. After some time, money becomes more plentiful, and is to be had at 4 per cent. D, therefore, naturally gives notice that, from a certain day, he will pay only 4 per cent.; and C is obliged either to accept the lower rate or to take up the principal—D, in the latter case, borrowing (if necessary) £1,000 from some body else, at 4 per cent., in order to pay C.

At present, every 20 shillings raised in taxes are spent in this way:—

	s.	*d.*
Expenses of Army, Navy, &c., . . .	7	9
Civil Services, ...	4	2
Interest of National Debt, . . .	8	1

So that we shall be very near the truth in saying that *two-fifths* of the amount realized by taxation—*i.e.*, £2 out of every £5—go in payment of the interest of the National Debt.

188. Stock can, at any time, be converted into ready-money—the holders being at liberty to transfer their claims to others, and there being always a number of persons anxious to purchase such claims.

The price of stock, like the price of any thing else, is a matter of "supply and demand:" stock is dear when the buyers are numerous, compared to the sellers; and *vice versa.* The value of stock, however, is peculiarly sensitive to political and other influences, and is even affected by mere rumours—often circulated by unscrupulous speculators. A general panic —occasioned by the prospect of war, or by any other circumstance tending to diminish public confidence in the stability of the State—would at once lower the price; inasmuch as an unusually large quantity of stock would then be offered for sale, whilst the number of buyers would be unusually small. Other circumstances being the same, stock is cheap when trade is flourishing; and *vice versa*: because, as a rule, capitalists do not purchase stock—which yields a comparatively small percentage—so long as trade is sufficiently brisk to enable them to employ their money more profitably.

189. Stock is said to be "at" PAR when its actual and its nominal value are equal in amount; "above" PAR, or *at a premium,* when its actual exceeds its nominal value; and "below" PAR, or *at a discount,* when its actual is less than its nominal value.

Thus, when £100 of it would realize, in *cash,*

£100, } stock is said to be { AT par;
£103, } stock is said to be { 3 ABOVE par, or at 3 *premium*;
£94, } stock is said to be { 6 BELOW par, or at 6 *discount.*

190. The stock most familiar to the general public is that which bears interest at 3 per cent. Of this stock—which, at a rough estimate, may be said to constitute *seven-eighths* of the National Debt—there are three varieties: (*a*) CONSOLIDATED Three-per-Cents—better known by the contracted name CONSOLS;* (*b*) REDUCED Three-per-Cents; and (*c*) NEW Three-per-Cents.

The first two varieties date from the year 1751, when (*a*) all the 3-per-cent. stocks previously kept separate were "consolidated" into one; and (*b*) when, also, the higher rates of interest, theretofore paid upon other portions of the National Debt, were "reduced" to 3 per cent. (*c*) The stock known as "New" Three-per-Cents is of recent creation—dating from the year 1854.

191. Stockholders† are paid their DIVIDENDS—that is, their respective shares of the interest of the National Debt—half-yearly; but the dividend-days are different for different kinds of stock. For Consols, the dividend-days are—5th of January and 5th of July; for both Reduced Threes and New Threes—5th of April and 10th of October.‡

So that if his money were invested—one-half, say, in Consols, and one-half in either Reduced Threes§ or New Threes, a stockholder would receive his dividend in quarterly instalments.

192. In Great Britain, the dividends are paid at the Bank of England; in Ireland, at the Bank of Ireland—those Banks acting in the matter as Government agents, and being, of course, remunerated for their services.

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* This stock is equal in amount to about *one-half* of the National Debt.

† The *average* number of stockholders is estimated at a quarter of a million (250,000).

‡ The dividends fall *due* upon those days, but are not *payable* until three days after.

§ Dealings in this stock, which amounts to upwards of £115,000,000, may be said to be confined to England.

193. The price of stock, when not an exact number of pounds, is expressed in pounds and a *fraction* of a pound—the denominator of the fraction being always 2, 4, or 8; and the £ (for "pounds") being dispensed with.

Thus, if the prices realized, in the course of a day, by £100 of Consols were £91$\frac{7}{8}$, £91$\frac{3}{4}$, and £91$\frac{1}{2}$, respectively, the fact would be announced in this way: "Consols...91$\frac{7}{8}$ $\frac{3}{4}$ $\frac{1}{2}$."

Three-per-cent. stock fetched its highest price, 107, in June, 1737; and its lowest price, 47$\frac{1}{4}$, in August, 1798. Now, the price usually ranges from 90 to 96.

194. Purchases and sales of stock are effected through a class of persons called *Stockbrokers*, who are licensed by the Lords of the Treasury. A stockbroker's fee is $\frac{1}{8}$ per cent.—that is, 2*s*. 6*d*. for every £100 of stock which he either buys or sells. So that when a transfer of stock takes place, the buyer pays $\frac{1}{8}$ per cent. more, and the seller receives $\frac{1}{8}$ per cent. less, than the quoted price.

When, for instance, the quoted price is 91$\frac{1}{2}$, £100 of stock costs the buyer £91$\frac{1}{2}$+£$\frac{1}{8}$=£91$\frac{5}{8}$, and realizes to the seller only £91$\frac{1}{2}$−£$\frac{1}{8}$=£91$\frac{3}{8}$.

195. In London, a person who buys stock becomes entitled to any interest that may be due on it at the time of purchase—such interest being included in the quoted price. In Dublin, however, the quoted price does *not* include the interest, for which a separate charge is made. In this circumstance—considered in connexion with the fact that the half-year's dividend is invariably paid to the person actually in possession of the stock on dividend-day—we have an explanation of two apparent anomalies, namely: (*a*) that the Funds are usually a shade higher in London than in Dublin; and (*b*) that, in London, but not in Dublin, the price of stock (in the absence of disturbing influences) becomes somewhat lower immediately after payment of a dividend, and then

gradually advances as the next dividend-day approaches.

196. When a person buys stock, and afterwards sells it, he generally gains or loses by the transaction,*—owing to the almost incessant fluctuations in the price.

This circumstance, as may be supposed, gives rise to a vast deal of speculation. When, for instance, £100 of stock is bought at 93 and sold at 95, a gain of £1¾ is realized—the difference between £93⅛, the price actually paid, and £94⅞, the price actually received. On the other hand, when £100 of stock is bought at 94½ and sold at 92¼, a loss of £2½ is sustained—the difference between £94⅝, the price actually paid, and £92⅛, the price actually received.

197. Even when the quoted price is exactly the same on the day of sale as on the day of purchase, a loss of 5*s.* is sustained on every £100 of stock dealt in.*

Thus, if the quoted price upon both occasions were 92⅜, the price actually paid for £100 of stock would be £92½, whilst the price actually received would be £92¼.

Amongst *Stockjobbers*—as mere speculators in the Funds are called—a common practice is this: A undertakes to sell, and B engages to buy, £1,000 (say) of stock, on a certain future day, at the price at which the stock is quoted on the day of the contract—94, for example. A has really no stock to sell; but he believes that, on or before the day agreed upon, the Funds will have fallen to 92 or 91, and that he will thus have an opportunity of "making money," by purchasing £1,000 of stock from a third person, at 92 or 91, and transferring it to B at 94. On the other hand, B enters into the engagement in the expectation that the Funds will have risen to 95 or 96 on or before the appointed day, and that, accordingly, *he* will be able to make money by selling to somebody else, at 95 or 96, the stock which A is to transfer to him (B) at 94.

On the Stock Exchange,† persons who enter into such bar-

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* Any interest received in the interval being, of course, left out of consideration.

† The place where stockbrokers meet for the transaction of business is called the *Stock Exchange*, but is more commonly known by the shorter name '*Change*.

gains—"time" bargains, as they are termed—are known as *Bears* and *Bulls*: a "Bear" being a person who, like A, engages to sell stock, in expectation of a fall in the price; and a "Bull" being a person who, like B, undertakes to buy, in anticipation of a rise. It is hardly necessary to observe that a contract between a Bear and a Bull is neither more nor less than a *wager* as to what the price of a certain quantity of stock will be at a future time—a wager, however, differing from ordinary ones in this, that the amount is undetermined until the time for payment arrives.

Stockjobbing transactions are settled monthly, upon a day fixed by the stockbrokers, and known as "*Account*-day." Dealings in stock which is not to be "delivered" until Account-day are spoken of as "For the Account." All other dealings are "For Present Delivery." As a general rule, the transactions of stockjobbers give rise to no actual transfer of stock; the Bear merely paying the Bull, or receiving from him, (as the case may be,) the difference between the values of the stock on the day of the contract and on Account-day. The amounts of such differences may be regarded as "debts of honour," their payment not being enforceable by law; but a defaulter is severely punished by being stigmatized as a *Lame Duck*, and being prevented from ever afterwards acting the part of either Bear or Bull.

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198. As a financial term, "stock" does not necessarily mean *Government* stock—being applied also to the capital of a commercial company. BANK stock, RAILWAY stock, &c. are illustrations of this: such stock usually consists of a number of SHARES, the holders of which are called *shareholders*.

Thus, when a number of capitalists, about to establish a bank or construct a railway, find that (say) £500,000 will be necessary, they subscribe the amount in 5,000 shares of £100 each, or in 10,000 shares of £50 each, or in 50,000 shares of £10 each, &c.; one person taking, perhaps, 100 shares, another 50, another 20, another 5, and so on. Sometimes the "promoters"—*i.e.*, the projectors—of the speculation contribute only a portion of the required capital, and invite the public to contribute the remainder.

When "allotted" a number of shares in a new undertaking, a person usually pays, at the time, only an instalment of the amount for which he becomes responsible; other instalments being subsequently paid, according as "calls" are made by the directors. In some instances, however, it is found that the full

amount is not required for the purposes of the undertaking; and thus it is that the *subscribed* capital of a company is sometimes a good deal in excess of the *paid-up* capital. If, for example, the stock of a company consisted (nominally) of 5,000 shares of £100 each, and the sum paid on each share were £80, the "subscribed" capital would be (5,000 × 100 =) £500,000, whilst the "paid-up" capital would be only (5,000 × 80 =) £400,000.

199. As to their liabilities, some companies are LIMITED, and some UNLIMITED. If a "limited-liability" company failed, the shareholders would not be legally responsible for more than their respective portions of the subscribed capital; but should an "unlimited" company become insolvent, the shareholders would all be liable—collectively and individually—for the full amount of the debts. A limited-liability company is known by its having the word "Limited" after its name: thus—"The Munster Bank, *Limited.*"

200. Railway and other shares are very extensively dealt in, and are quoted *at*, *above*, or *below* par—according to the dividends they are likely to realize. When buying or selling shares, a stockbroker charges, in most cases, a fee of $\frac{1}{4}$ per cent.; *i.e.*, 5*s.* on every £100.*

[In the following examples—which, it will be seen, belong to Proportion—the price mentioned, as that of £100 of stock, is supposed in every instance to be the price *actually* paid or received; in other words, the "quoted" price *plus* or *minus* (as the case may be) the fee charged by the broker:—]

* "Notwithstanding the low interest paid by the Funds, it is by far the best investment that any ordinary person can select for the money which he does not employ in his own business. Many investments promise a higher rate of interest, but they are suitable only for persons with large capital, who make it their special business to know what investments offer sufficient security. But the man who has a few hundred, or even a few thousand pounds to dispose of, if he selects any other investment will generally find in the end that, owing to the failure of public companies, law expenses, bad debts, and other losses, he would have been much richer if he had invested his money in the Funds."—*Judge Longfield.*

EXAMPLE I.—How much stock can be bought, at $89\frac{5}{8}$, for £750?

Worded somewhat differently, the question is this: If £$89\frac{5}{8}$ (cash) will purchase £100 of stock, what quantity of stock will £750 (cash) purchase? The proportion is—

$$\begin{array}{cccc} \text{(Cash.)} & \text{(Cash.)} & \text{(Stock.)} & \text{(Stock.)} \\ £89\tfrac{5}{8} & : £750 & : : £100 & : a \end{array}$$

$$a = (750 \times 100 \div 89\tfrac{5}{8} =) £836\ 16s.\ 5d. \text{ nearly.}$$

EXAMPLE II.—If £365 of stock were sold at $91\frac{1}{8}$, what would the seller receive in ready-money?

In other words: If £100 of stock would realize £$91\frac{1}{8}$ (cash), how much (cash) would £365 of stock realize? The proportion is—

$$\begin{array}{cccc} \text{(Stock.)} & \text{(Stock.)} & \text{(Cash.)} & \text{(Cash.)} \\ £100 & : £365 & : : £91\tfrac{1}{8} & : a \end{array}$$

$$a = (365 \times 91\tfrac{1}{8} \div 100 =)\ £332\ 12s.\ 1d.$$

EXAMPLE III.—What per-centage does a person receive on money which he invests in Consols at 90?

On £90 (cash) he receives the interest of £100 of Consols—that is, £3 a year; therefore, on £100 (cash) the yearly interest is the fourth term of the proportion—

$$\begin{array}{cccc} \text{(Prin.)} & \text{(Prin.)} & \text{(Int.)} & \text{(Int.)} \\ £90 & : £100 & : : £3 & : a \end{array}$$

$$a = (100 \times 3 \div 90 =)\ £3\tfrac{1}{3}, \text{ or } £3\ 6s.\ 8d.$$

EXAMPLE IV.—Which is the more profitable investment (other circumstances being the same)—stock bought at 230, and paying 10 per cent.;* or stock bought at 120, and paying 6 per cent.?*

When 10-per-cent stock is bought at 230, the purchaser receives £10 a-year (interest) on £230 (principal). On £100 (principal), therefore, he receives £$4\frac{8}{23}$ a-year (interest), or a little more than £$4\frac{1}{3}$:

$$\begin{array}{cccc} \text{(Prin.)} & \text{(Prin.)} & \text{(Int.)} & \text{(Int.)} \\ £230 & : £100 & : : £10 & : a \end{array}$$

$$a = (100 \times 10 \div 230 =)\ £4\tfrac{8}{23}.$$

Again: When 6-per-cent. stock is bought at 120, the pur-

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* That is, on the paid-up capital.

chaser receives £6 a-year (interest) on £120 (principal). On £100 (principal), therefore, he receives £5 a-year (interest):

(Prin.) (Prin.) (Int.) (Int.)

$$£120 : £100 :: £6 : a$$
$$a = (100 \times 6 \div 120 =) \ £5.$$

So that the *actual* per-centage is greater in the second case than in the first by nearly £$\frac{2}{3}$.

A comparison could also be instituted in this way: The yearly interest of £230 being £10, what is the yearly interest of £120—at the same rate? The proportion is—

(Prin.) (Prin.) (Int.) (Int.)

$$£230 : £120 :: £10 : a$$
$$a = (120 \times 10 \div 230 =) \ £5\tfrac{5}{23}.$$

On £120, therefore, the yearly interest would be (£6—£$5\frac{5}{23}$=) £$\frac{18}{23}$ more in the 6-per-cent stock than in the 10-per-cent. stock; so that £100 would produce nearly £$\frac{2}{3}$ a-year more in the one stock than in the other:

$$£120 : £100 :: £\tfrac{18}{23} : a$$
$$a = (100 \times \tfrac{18}{23} \div 120 =) \ £\tfrac{15}{23}, \text{ or } £\tfrac{2}{3} \text{ nearly.}$$

EXAMPLE V.—What quantity of New Threes, at $90\frac{1}{2}$, could be obtained for £1,500 of $3\frac{1}{2}$ per-cent. stock, at $96\frac{1}{4}$?

The lower the price of stock, the larger the quantity required to realize a certain sum of money, and *vice versa*. As *small*, therefore, as $90\frac{1}{2}$ is, compared to $96\frac{1}{4}$, so *large* must be the quantity of New Threes, compared to that of the $3\frac{1}{2}$-per-cent. stock—in other words, so *small* must be the quantity of the latter stock compared to that of the former. We thus have the (inverse) proportion—

$$90\tfrac{1}{2} : 96\tfrac{1}{4} :: 1{,}500 : a$$
$$a = (1{,}500 \times 96\tfrac{1}{4} \div 90\tfrac{1}{2} =) \ £1{,}595 \ 6s. \ 1d. \text{ nearly.}$$

Here is a more round-about solution: £1,500 of stock, if sold at $96\frac{1}{4}$, would realize, in cash, £$1{,}443\frac{3}{4}$:

(Stock.) (Stock.) (Cash.) (Cash.)

$$£100 : £1{,}500 :: £96\tfrac{1}{4} : a$$
$$a = (1{,}500 \times 96\tfrac{1}{4} \div 100 =) \ £1{,}443\tfrac{3}{4}.$$

And £$1{,}443\frac{3}{4}$, if invested in New Threes at $90\frac{1}{2}$, would purchase £1,595 6s. 1d. (nearly) of that stock:

(Cash.) (Cash.) (Stock.) (Stock.)

$$£90\tfrac{1}{2} : £1{,}443\tfrac{3}{4} :: £100 : a$$
$$a = (1{,}443\tfrac{3}{4} \times 100 \div 90\tfrac{1}{2} =) \ £1{,}595 \ 6s. \ 1d. \text{ nearly.}$$

PROFIT AND LOSS.

201. Under this head, we apply our knowledge of Proportion to questions relating to the gains and losses of people in trade. Such gains and losses are usually expressed as so much PER CENT.—that is, (*at the rate of*) so much on £100, cost-price.

When an article which cost £5 is sold for £6, there is a gain of £1; and when an article which cost £20 is sold for £22, there is a gain of £2. The first of these gains, although *absolutely* less, is *relatively* greater, than the second; a gain of £1 on an outlay of £5 being at the same rate as a gain of (not £2, but) £4 on an outlay of £20:

$$£5 : £20 :: £1 : a; \ a = (20 \div 5 =) £4.$$

Again: when an article which cost £1 is sold for 19*s.* 8*d.*, there is a loss of 4*d.*; and when an article which cost 3*s.* 4*d.* is sold for 3*s.*, there is a loss of 4*d.* also. The first loss, however, although absolutely the same as the second, is relatively smaller; a loss of 4*d.* on an outlay of £1 being at the same rate as a loss of *two-thirds* of 1*d.* on an outlay of 3*s.* 4*d.*:

$$240d. : 40d. :: 4d. : a; \ a = (40 \times 4 \div 240 =) \tfrac{2}{3}d.$$

So that, to form a just estimate of gains and losses, we must, in every case, take the cost-price into account; and the adoption of £100, as a standard cost-price, enables us at once to contrast one gain with another, or one loss with another. The following examples explain themselves:—

EXAMPLE I.—A draper bought a piece of cloth, 50 yards long, for £27 10*s.*, and realized by the sale of it a gain of £3 15*s.*; what was the selling-price per yard?

Adding the gain to the cost-price, we find that the selling-price of the piece was (£27 10*s.* + £3 15*s.* =) £31 5*s.*; so that the selling-price per yard was £31 5*s.* ÷ 50 = 12*s.* 6*d.*

EXAMPLE II.—A quantity of merchandise which cost £640 was sold for £725; what was the gain per cent.?

On £640 the gain was (£725 − £640 =) £85; therefore, to find the gain on £100, we say—

(Cost-price.)		(Cost-price.)		(Gain.)		(Gain.)
£640	:	£100	::	£85	:	*a*

$$a = (85 \times 100 \div 640 =) £13\ 5s.\ 7\tfrac{1}{2}d., \text{ the gain per cent.}$$

EXAMPLE III.—A person bought a quantity of hay for £250, and sold it for £235; how much per cent. did he lose?

On £250 he lost (£250—£235=) £15; therefore, to find his loss on £100, we say—

(Cost-price.) (Cost-price.) (Loss.) (Loss.)

£250 : £100 :: £15 : a

a=(15 × 100÷250=) £6, the loss per cent.

EXAMPLE IV.—A quantity of tea, which cost £365, was sold at a profit of 20 per cent.; what was the selling-price?

Every £100, cost-price, realized (£100+£20=) £120, selling-price; therefore, to find the selling-price realized by £365, cost-price, we say—

(Cost-price.) (Cost-price.) (Selling-price.) (Selling-price.)

£100 : £365 :: £120 : a

a=(365 × 120÷100=) £438, the required selling-price.

EXAMPLE V.—An estate, which cost £3,000, was sold at a loss of 15 per cent.; what was the selling-price?

Every £100, cost-price, realized only (£100—£15=) £85, selling-price; therefore, to find the selling-price realized by £3,000, cost-price, we say—

(Cost-price.) (Cost-price.) (Selling-price.) (Selling-price.)

£100 : £3,000 :: £85 : a

a=(3,000 × 85÷100=) £2,550, the required selling-price.

EXAMPLE VI.—A merchant sold a quantity of rice for £585, and gained 12½ per cent.; what did the rice cost him?

Every £100, cost-price, produced (£100+£12½=) £112½, selling-price; therefore, to find the cost-price which produced £585, selling-price, we say—

(Selling-price.) (Selling-price.) (Cost-price.) (Cost-price.)

£112½ : £585 :: £100 : a

a=(585 × 100÷112½=) £520, the required cost-price.

EXAMPLE VII.—A grazier sold a flock of 200 sheep for £270, and lost 10 per cent.; how much—on an average—did each sheep cost him?

Every £100, cost-price, produced only (£100—£10=) £90, selling-price; therefore, to find what cost-price produced £270, selling-price, we say—

(Selling-price.) (Selling-price.) (Cost-price.) (Cost-price.)

£90 : £270 :: £100 : a

a=(270 × 100÷90=) £300, the cost-price of the flock.

On an average, therefore, each sheep cost the grazier £300÷200=£1 10*s*.

DIVISION INTO PROPORTIONAL PARTS.

202. Under this head, we employ our knowledge of Proportion in dividing a number into two or more parts proportional to other numbers.

Example I.—Divide £40 between A and B, so that A's share shall bear to B's the ratio of 3 to 2.

Out of every (£3+£2=) £5 in the given amount, £3 must be given to A, and £2 to B. Consequently, as £40 contains 8 sums of £5 each, A must get 8 sums of £3 each, and B 8 sums of £2 each; in other words, £3 must bear to A's share, and £2 to B's share, the ratio which £5 bears to £40. We thus have the proportions—

£5 : £40 :: £3 : A's share.
£5 : £40 :: £2 : B's „

Or*—

5 : 3 :: £40 : A's share (£24.)
5 : 2 :: £40 : B's „ (£16.)

Example II.—Divide an estate of 2,160 acres between A, B, and C, so that A's share shall bear to B's the ratio of 7 to 8, and B's share to C's the ratio of 8 to 9.

Out of every (7+8+9=) 24 acres, 7 must be given to A, 8 to B, and 9 to C. As large, therefore, as the estate is, compared to 24 acres, so large must be A's share compared to 7 acres, B's share compared to 8 acres, and C's share compared to 9 acres. We thus have—

Acres.	Acres.	Acres.	
24 :	2160 ::	7 :	A's share.
24 :	2160 ::	8 :	B's „
24 :	2160 ::	9 :	C's „

Or*—

		Acres.	Acres.
24 :	7 ::	2160 :	A's share (630.)
24 :	8 ::	2160 :	B's „ (720.)
24 :	9 ::	2160 :	C's „ (810.)

Example III.—Divide 1,000 into four parts proportional to the fractions $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, and $\frac{1}{6}$: that is, into four such parts that the first shall bear to the second the ratio of $\frac{1}{3}$ to $\frac{1}{4}$; the second to the third, the ratio of $\frac{1}{4}$ to $\frac{1}{5}$; and the third to the fourth, the ratio of $\frac{1}{5}$ to $\frac{1}{6}$.

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* By Alternation.

The preceding examples suggest the following solution:—

$$\tfrac{1}{3}+\tfrac{1}{4}+\tfrac{1}{5}+\tfrac{1}{6}=\tfrac{19}{20};$$

$\frac{19}{20} : \frac{1}{3} :: 1{,}000 : a$; $a=350\frac{50}{57}$, the first part

$\frac{19}{20} : \frac{1}{4} :: 1{,}000 : b$; $b=263\frac{3}{19}$, ,, second ,,

$\frac{19}{20} : \frac{1}{5} :: 1{,}000 : c$; $c=210\frac{10}{19}$, ,, third ,,

$\frac{19}{20} : \frac{1}{6} :: 1{,}000 : d$; $d=175\frac{25}{57}$, ,, fourth ,,

This solution is easily explained. It is obvious that, if the fractions $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, and $\frac{1}{6}$ were all multiplied by the *same* number, the resulting products would be proportional to the fractions (p. 165); and it is equally obvious that, if their sum were exactly 1,000, those products would be the proportional parts required. So that, m being put for the multiplier which would give products amounting in the aggregate to 1,000, we have $\frac{1}{3}\times m+\frac{1}{4}\times m+\frac{1}{5}\times m+\frac{1}{6}\times m=1{,}000$; $(\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6})\times m=1{,}000$; $\frac{19}{20}\times m=1{,}000$; $m=\frac{1{,}000}{\frac{19}{20}}$. The proportional parts being represented by a, b, c, and d, respectively, we also have—

$$a=\tfrac{1}{3}\times m=\tfrac{1}{3}\times\frac{1{,}000}{\frac{19}{20}};\ a\times\tfrac{19}{20}=\tfrac{1}{3}\times 1{,}000$$

$$b=\tfrac{1}{4}\times m=\tfrac{1}{4}\times\frac{1{,}000}{\frac{19}{20}};\ b\times\tfrac{19}{20}=\tfrac{1}{4}\times 1{,}000$$

$$c=\tfrac{1}{5}\times m=\tfrac{1}{5}\times\frac{1{,}000}{\frac{19}{20}};\ c\times\tfrac{19}{20}=\tfrac{1}{5}\times 1{,}000$$

$$d=\tfrac{1}{6}\times m=\tfrac{1}{6}\times\frac{1{,}000}{\frac{19}{20}};\ d\times\tfrac{19}{20}=\tfrac{1}{6}\times 1{,}000$$

Hence the above proportions (§ 167).

203. To divide a number into Proportional Parts: Set down, as the second terms of so many proportions, the numbers to which the parts are to be proportional; make the sum of these numbers the first term, in every case; and write, as the third term of each proportion, the number to be divided. The fourth terms will be the proportional parts required.

NOTE.—We verify the work by adding the proportional parts together, and finding their sum equal to the number proposed for division. Thus, £24+£16=£40 (Ex. I.); 630 acres+720 acres+810 acres=2,160 acres (Ex. II.); $350\frac{50}{57}+263\frac{3}{19}+210\frac{10}{19}+175\frac{25}{57}=1{,}000$ (Ex. III.)

FELLOWSHIP OR PARTNERSHIP.

204. FELLOWSHIP—or, as it is sometimes termed, PARTNERSHIP—deals with the individual gains or losses of the partners in a mercantile company, when the total gain or loss of the company is known.

205. Fellowship is said to be SIMPLE or COMPOUND—according as the partners' respective portions of the capital have been in the business for the *same* length of time, or for *unequal* periods.

From the examples which follow, it will be seen that Simple Fellowship is nothing more than Division into Proportional Parts; and that the conversion of "Compound" into "Simple" Fellowship is merely a matter of Multiplication.

SIMPLE FELLOWSHIP.

EXAMPLE I.—A and B enter into partnership, and gain £96; A's share of the capital is £350, and B's share £250; how much of the gain ought each partner to receive?

It is evident that, if A's portion of the capital were equal to B's portion, the partners would each be entitled to one-half of the profit; that, if A's portion of the capital were twice as large as B's portion, A should receive twice as much of the profit as B; and so on. We have therefore to divide £96 into two parts proportional to the numbers 350 and 250:—

$350+250=600$;
$600 : 350 :: £96 : a$; $a=£56$, A's gain.
$600 : 250 :: £96 : b$; $b=£40$, B's ,,

EXAMPLE II.—Three persons, A, B, and C, enter into partnership, and lose £365; A's share of the capital is £820, B's share £750, and C's share £640; how much of the loss ought each person to sustain?

Here we have to divide £365 into three parts proportional to the numbers 820, 750, and 640:—

$820+750+640=2210$;

		£	s.	d.	
$2210 : 820 :: £365 : a$;	$a=$	135	8	$7\frac{1}{4}$,	A's loss.
$2210 : 750 :: £365 : b$;	$b=$	123	17	$4\frac{1}{2}$,	B's ,,
$2210 : 640 :: £365 : c$;	$c=$	105	14	$0\frac{1}{4}$,	C's ,,

EXAMPLE III.—A bankrupt, whose assets amount to £500, has four creditors—A, B, C, and D—to whom he owes £340, £230, £200, and £150, respectively; what portion of the assets ought each creditor to receive?

Dividing £500 into four sums proportional to the numbers 340, 230, 200, and 150, we have—

$$340+230+200+150=920;$$

			£	s.	d.	
920 : 340 ::	£500 : a;	$a=$	184	15	$7\frac{3}{4}$,	A's portion.
920 : 230 ::	500 : b;	$b=$	125	0	0,	B's "
920 : 200 ::	500 : c;	$c=$	108	13	11,	C's "
920 : 150 ::	500 : d;	$d=$	81	10	$5\frac{1}{4}$,	D's "

NOTE.—The last example, although worked in the same way as the preceding ones, can hardly be said to belong to Fellowship.

COMPOUND FELLOWSHIP.

EXAMPLE I.—Two persons, A and B, were in partnership, and gained £900; A's capital, £2,500, was only 7 months in the business, whilst B's capital, £1,800, was in the business for 11 months; how ought the gain to be divided?

The use of £2,500 for 7 months is equivalent to the use of 7 times £2,500, or of £17,500, for one month; and the use of £1,800 for 11 months is equivalent to the use of 11 times £1,800, or of £19,800, for one month. So that the question can be put in this way, and made one in Simple Fellowship: A and B were a month in partnership, and gained £900; A's capital was £17,500, and B's £19,800; how ought the gain to be divided? Here, therefore, is the solution:—

$$\begin{aligned} 2{,}500\times 7&=17{,}500\\ 1{,}800\times 11&=19{,}800\\ \hline &37{,}300 \end{aligned}$$

			£	s.	d.	
37,300 : 17,500 ::	£900 : a;	$a=$	422	5	$0\frac{1}{2}$,	A's gain.
37,300 : 19,800 ::	900 : b;	$b=$	477	14	$11\frac{1}{2}$,	B's "

EXAMPLE II.—A, B, and C were in partnership, and lost £1,500; A had £5,000 in the business for 6 weeks, B £4,000 for 7 weeks, and C £3,000 for 9 weeks; how much of the loss ought each partner to sustain?

Here we have—

£5,000 for 6 weeks=(£5,000 × 6=)£30,000 for one week.
£4,000 " 7 " =(£4,000 × 7=)£28,000 " " "
£3,000 " 9 " =(£3,000 × 9=)£27,000 " " "

We therefore divide £1,500 into three parts proportional to the numbers 30,000, 28,000, and 27,000—or to the numbers 30, 28, and 27:

$$30+28+27=85;$$

	£		£	s.	d.	
85 : 30 ::	1,500 : a;	a=	529	8	$2\frac{3}{4}$,	A's loss.
85 : 28 ::	1,500 : b;	b=	494	2	$4\frac{1}{4}$,	B's ,,
85 : 27 ::	1,500 : c;	c=	476	9	5	C's ,,

EXCHANGE.

206. A knowledge of EXCHANGE enables us to find how much of the money of a country is equivalent to a given amount of the money of another country.

207. The *intrinsic* value of the money of a country compared to the money of another country—the comparison being based upon the weight and fineness of the two sets of coins—is called the "PAR of Exchange" between those countries.

A distinction is sometimes drawn between the *intrinsic* and the *commercial* par of Exchange. The "intrinsic" par is calculated on the supposition that the market value of pure gold is everywhere the same, and bears a constant ratio to the market value of pure silver. Owing, however, to certain causes which cannot be entered into here, the market values of the precious metals are occasionally a little higher in one country than in another; and this circumstance enters into the calculation of the "commercial" par. When the word "commercial" is not mentioned, the par of Exchange between two countries is always understood to mean the *intrinsic* par, which, it is hardly necessary to observe, remains the same so long as no alteration is made in the currency of either country.

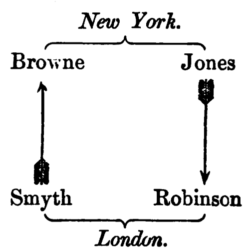
208. The amount of the money of one country (sometimes more, and sometimes less, according to the circumstances of trade) that is actually given—at any particular time—for a certain amount of the

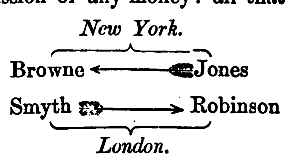
money of another country, is termed the "COURSE of Exchange"—at that time—between those countries.

Thus, whilst the intrinsic value, in French money, of £1 is 25 francs 22 centimes, £1 may be convertible into so much as 25 francs 50 centimes at one time, and into only 25 francs 10 centimes at another. When the amount of French money that can be obtained for £1 is exactly 25 francs 22 centimes, the course of Exchange with France is said to be "at" PAR; and the course of Exchange is spoken of as "above" or "below" PAR—according as more or less than 25 francs 22 centimes can be got for £1.

209. In mercantile transactions between different countries, payments are usually made by means of bills of exchange—*i.e.*, "foreign" bills (see p. 203).

Let us suppose that Mr. Smyth of London owes Mr. Browne of New York £500, and that Mr. Jones of New York owes Mr. Robinson of London an equal amount. At first sight, these two transactions would appear to involve the transmission of £500 from London to New York, and of £500 more from New York to London. A little reflection, however, makes it evident that both accounts can be settled without the transmission of any money: all that is necessary being—an arrangement by means of which Browne is paid by Jones instead of by Smyth, and Robinson is paid by Smyth instead of by Jones.





Such an arrangement, however, does not necessarily suppose any acquaintanceship either between Smyth and Jones or between Browne and Robinson. In places like London and New York, foreign bills are bought and sold like any other description of property; buyers and sellers being, in most cases, brought together through the agency of a class of persons called *Billbrokers*. In the case under consideration, what occurs is this: (*a*) Jones gives Browne £500 (or thereabouts)

for a £500 bill *on* Smyth, *in favour of* the person named by Jones—that is, Robinson; this bill Jones then sends to Robinson, who presents it to Smyth for payment. Or (*b*) Smyth gives Robinson £500 (or thereabouts) for a £500 bill *on* Jones, *in favour of* the person named by Smyth—that is, Browne; this bill Smyth then sends to Browne, who presents it to Jones for payment.

So that if the debts due by London to New York and those due by New York to London happened to be equal in amount, the accounts between the two places could all be settled by means of bills of exchange: the demand, in each place, for bills "on"—*i.e.*, payable in—the other place would be exactly equal to the supply, and the course of exchange would be at par. But if London owed New York £80,000, whilst New York owed London only £70,000, (in other words, if the value of the imports from New York exceeded that of the exports to it by £10,000,) London would be obliged to send New York £10,000 in money: bills on New York would then fetch a higher price than usual in London, because of the increased demand for them—compared to the supply; and the course of exchange would be *against* London, and *in favour of* New York. On the other hand, if New York owed London £130,000, whilst London owed New York only £100,000, New York would be obliged to send London £30,000 in money: bills on London would then fetch a higher price than usual in New York; and the course of exchange would be *against* New York, and *in favour of* London.*

In no case, however, can the price paid for a foreign bill exceed the amount of the bill by a larger sum than would be charged for the transmission and insurance of the amount, in gold or silver.

It is obvious that when specie or bullion to the amount of £500, for instance, can be sent from London to New York, and insured against risk, for 1 per cent.,—that is, for £5,—a £500 bill on New York cannot fetch a higher price in London than £505.

210. Foreign bills are sometimes transmitted from one country to another—not directly, but (so to speak) circuitously; that is, through one or more other countries.

* In order to appreciate the expressions "against" and "in favour of," we must remember that a certain amount of loss is sustained by a body of merchants who, for goods received, are obliged to give money instead of merchandise; and that the withdrawal of money from a country is regarded as more or less detrimental to the country at large.

Thus, if bills on Paris were at a premium of $\frac{1}{4}$ per cent. in St. Petersburg, and bills on London at a premium of 1 per cent. in Paris, whilst bills on London were at a premium of only $\frac{5}{8}$ per cent. in St. Petersburg, a London merchant who wanted to send a bill on London to St. Petersburg would find it more advantageous to transmit the bill through Paris than to make a direct transmission. For the sake of simplicity, let us suppose that the amount of the bill about to be transmitted is £100. If sent directly, this bill would realize only £100$\frac{5}{8}$, or £100 12*s.* 6*d.*, in St. Petersburg; whereas, if sent through a correspondent in Paris,—who would sell or exchange it for £101 of Paris "paper," and then send this paper to St. Petersburg,—the bill would realize £101 5*s.*

211. When a foreign bill is transmitted circuitously—instead of directly—from one country to another, the price ultimately realized by the bill is termed an "*arbitrated*" rate of Exchange between the two countries; and the calculation of such rates, from the necessary data, belongs to what is called "ARBITRATION of Exchange."

In the case just supposed, £100 of British money—although, according to the "course" of Exchange, equivalent to only £100 12*s.* 6*d.* of Russian money—can, by an "arbitrated" Exchange, be converted into £101 5*s.* Russian currency.

212. Arbitration of Exchange is said to be SIMPLE or COMPOUND—according as the number of places to be taken into account is three or more than three; in other words—according as the number of "intermediate" places is one or more than one.

EXAMPLE I. (*Par* of Exchange.)—Find the par of Exchange between London and Paris: the data being—that the franc (the French standard of value) contains, out of 10 parts, 9 of pure silver and 1 of alloy; that 200 francs weigh a kilogramme (15,434 grains); that the napoleon, which contains, out of 10 parts, 9 of pure gold and 1 of alloy, is equivalent to 20 francs; that 155 napoleons (which are equivalent to 3,100 francs) weigh a kilogramme; that the mint price of the sovereign (the British standard of value), which contains, out of 12 parts, 11 of pure gold and 1 of alloy, is £3 17*s.* 10$\frac{1}{2}$*d.*, or 934·5 pence, an ounce Troy; and that the market value of English standard silver, which contains, out of 40 parts, 37 of pure silver and 3 of alloy, is (about) 5 shillings an ounce Troy.

Here we have—

£1	=	240*d.*
934·5*d.*	=	1 oz. English standard gold
12 oz. English standard gold	=	11 oz. pure gold
9 oz. pure gold	=	10 oz. French standard gold
1 oz. French standard gold	=	480 grs.
15,434 grs.	=	1 kilogramme
1 kilog. French standard gold	=	3,100 francs
? francs	=	£1

Hence, by the Chain Rule (p. 179), £1=(240×11×10×480×3,100÷934·5×12×9×15,434=)25·22 francs, or 25 francs 22 centimes; and this is the par of Exchange—the GOLD par, as it is sometimes termed—between London and Paris.

NOTE.—From a comparison of the standard silver of England with that of France, another par, called the *silver* par, can be obtained as follows:

£1	=	240*d.*
60*d.*	=	1 oz. English standard silver
40 oz. English standard silver	=	37 oz. pure silver
9 oz. pure silver	=	10 oz. French standard silver
1 oz. French standard silver	=	480 grs.
15,434 grs.	=	1 kilogramme
1 kilog. French standard silver	=	200 francs
? francs	=	£1

£1=(240 × 37 × 10 × 480 × 200÷60 × 40 × 9 × 15,434=) 25·57 francs, or 25 francs 57 centimes; and this is the silver par between London and Paris.

EXAMPLE II. (*Course* of Exchange.)—If the course of Exchange between England and France were 25 francs 46 centimes for £1, what amount of British money would be equivalent to 567 francs 38 centimes?

The answer is evidently the fourth term of the proportion—

francs.		francs.				
25·46	:	567·38	: :	£1	:	*a*

a=(567·38÷25·46=)£22·285, or £22 5*s.* 8½*d.*

EXAMPLE III. (*Course* of Exchange.)—If the course of Exchange between England and Russia were 3*s.* 3½*d.* for 1 rouble, what amount of Russian money would be equivalent to £162 11*s.* British?

The proportion is—

3*s.* 3½*d.* : £162 11*s.* : : 1 rouble : *a*; or

39·5*d.* : 39,012*d.* : : 1 rouble : *a*

a=(39,012÷39·5=)987·65 roubles (nearly), or 987 roubles 65 copecks [1 rouble=100 copecks].

NOTE.—In the courses of Exchange between England and other countries, a variable amount of foreign money is, in some cases, allowed for a fixed amount of British; whilst, in other cases, a variable amount of British is allowed for a fixed amount of foreign money. Thus, in the Exchange between England and France the "fixed" rate (£1) is given by England, and the "variable" rate by France; but in the Exchange between England and Russia the fixed rate (1 rouble) is given by Russia, and the variable rate—sometimes 3*s.* 2*d.*, sometimes 3*s.* 4*d.*, &c.—by England.

EXAMPLE IV. (*Simple* ARBITRATION of Exchange.)—The course of Exchange between London and Paris being 24 francs 85 centimes for £1, and between Paris and St. Petersburg 7 francs for 2 roubles, what is the arbitrated rate between London and St. Petersburg?

Here we have—

£1 =24·85 francs
7 francs = 2 roubles
? roubles=£1

£1=(24·85×2÷7=)7·1 roubles, or 7 roubles 10 copecks.

EXAMPLE V. (*Compound* ARBITRATION of Exchange.)—The course of Exchange between London and Frankfort being 11 florins 30 kreutzers [1 florin = 60 kreutzers] for £1, between Frankfort and Paris 4 florins 18 kreutzers for 9 francs, and between Paris and Lisbon 13 francs 20 centimes for 3 milrees, what is the arbitrated rate between London and Lisbon?

Here we have—

£1 = 11·5 florins
4·3 florins = 9 francs
13·2 francs = 3 milrees
? milrees =£1

£1=(11·5×9×3÷4·3×13·2=)5·47 milrees, or 5 milrees 470 rees [1 milree = 1,000 rees].

ALLIGATION.

213. Under this head come questions which relate to the mixing of different kinds of the same commodity—tea, sugar, wine, &c.; the object of such mixing being (*a*) to improve the quality of an inferior article, or (*b*) to make a superior article cheaper and more saleable.

The method of working exercises in ALLIGATION—exercises which, it may be observed, seldom occur in practice—will be understood from the following examples:

EXAMPLE I.—If 20 lbs. of tea worth 2*s*. 6*d*. a pound, 16 lbs. worth 3*s*. a pound, and 12 lbs. worth 3*s*. 4*d*. a pound were mixed together, how much a pound would the mixture be worth, on an average?

	lbs.		*s.*	*d.*			*s.*
Price of	20	@	2	6	per pound	=	50
,,	16	,,	3	0	,,	=	48
,,	12	,,	3	4	,,	=	40
,,	48					=	138

Average price of 1 lb. of the mixture = 138*s*. ÷ 48 = 2*s*. 10½*d*.

EXAMPLE II.—In what proportions does a grocer mix sugar worth 6*d*. a pound and sugar worth 4*d*. a pound, when he wants the mixture to be worth, on an average, 5¼*d*. a pound?

On every pound of 6-penny sugar in the mixture, the grocer loses (6*d*. − 5¼*d*.=) 3 farthings; whilst on every pound of 4-penny sugar he gains (5¼*d*. − 4*d*.=) 5 farthings. Consequently, as the total loss and the total gain must exactly counterbalance each other, the grocer, instead of mixing the two sugars in equal proportions, makes the 6-penny sugar as much larger in quantity than the 4-penny, as 5 farthings are larger in amount than 3 farthings. So that 5 lbs. of the dearer, and 3 lbs. of the cheaper sugar would form the required mixture; 5 times 3 farthings being the loss on the former, and 3 times 5 farthings the gain on the latter:

lbs.		*d.*			*d.*
5	@	6	per pound	=	30
3	,,	4	,,	=	12
8	,,	5¼	,,	=	42

In practice, the work would assume the following form—the numbers indicating the gain and the loss per pound being taken *cross-wise* for the required proportions:

$$5\tfrac{1}{4}d.\left\{\begin{array}{l} \overset{d.}{6};\ \textit{loss}\text{ per pound} = \overset{\text{farthings.}}{3} \\ 4;\ \textit{gain}\quad ,, \quad = 5 \end{array}\right. \quad \text{Ans.}\left\{\begin{array}{l} \overset{\text{lbs.}}{5}\ @\ \overset{d.}{6}\text{ per pound.} \\ 3\ ,,\ 4\quad ,, \end{array}\right.$$

NOTE.—If a particular quantity of the mixture were required, the number denoting the quantity should be divided into parts proportional to 5 and 3. Thus, in 4 lbs. of the mixture there would be $2\frac{1}{2}$ lbs. of the 6-penny and $1\frac{1}{2}$ lbs. of the 4-penny sugar; in 24 lbs. of the mixture, 15 lbs. of the 6-penny and 9 lbs. of the 4-penny sugar; &c.:—

lbs.	d.	d.	lbs.	d.	d.
$2\frac{1}{2}$	@ 6	= 15	15	@ 6	= 90
$1\frac{1}{2}$	,, 4	= 6	9	,, 4	= 36
4	,, $5\frac{1}{4}$	= 21	24	,, $5\frac{1}{4}$	= 126

EXAMPLE III.—In what proportions should a vintner mix four different kinds of sherry—worth, respectively, 15*s.*, 20*s.*, 30*s.*, and 32*s.* a gallon—in order to form a mixture worth, on an average, 24*s.* a gallon?

On every gallon of the—

$$\left.\begin{array}{l} \overset{s.}{15} \\ 20 \\ 30 \\ 32 \end{array}\right\}\text{ sherry, there would be a }\left\{\begin{array}{l} \textit{gain}\text{ of }\overset{s.}{9} \\ \textit{gain}\ ,,\ 4 \\ \textit{loss}\ ,,\ 6 \\ \textit{loss}\ ,,\ 8 \end{array}\right.$$

It is evident, therefore, after what has already been explained, that the gains would counterbalance the losses if (*a*) the mixture were made to contain 6 gallons of the first sherry for every 9 of the third, and 8 gallons of the second for every 4 of the fourth; or if (*b*) 8 gallons of the first were taken for every 9 of the fourth, and 6 gallons of the second for every 4 of the third.

According to the first arrangement, the gains and losses would stand thus:

$$\begin{array}{llll} & \text{galls.} & s. & & s. \\ \text{On} & 6 & \text{of the } 15 & \text{sherry, a } \textit{gain} \text{ of} & 6 \times 9 \\ ,, & 9 & ,,\ ,,\ 30 & ,,\quad ,,\ \textit{loss}\ ,, & 9 \times 6 \\ ,, & 8 & ,,\ ,,\ 20 & ,,\quad ,,\ \textit{gain}\ ,, & 8 \times 4 \\ ,, & 4 & ,,\ ,,\ 32 & ,,\quad ,,\ \textit{loss}\ ,, & 4 \times 8 \end{array}$$

According to the second arrangement, the gains and losses would be—

$$\begin{array}{lllll} & \text{galls.} & s. & & \\ \text{On} & 8 & \text{of the } 15 & \text{sherry, a } \textit{gain} \text{ of} & 8 \times 9 \\ ,, & 9 & ,,\ ,,\ 32 & ,,\quad ,,\ \textit{loss}\ ,, & 9 \times 8 \\ ,, & 6 & ,,\ ,,\ 20 & ,,\quad ,,\ \textit{gain}\ ,, & 6 \times 4 \\ ,, & 4 & ,,\ ,,\ 30 & ,,\quad ,,\ \text{loss}\ ,, & 4 \times 6 \end{array}$$

So that the wines could be mixed in either of two ways—corresponding to the two different ways in which the gains and the losses could be made to counterbalance one another :

$$24s. \begin{cases} \overset{s.}{15}; \textit{ gain} \text{ on } \overset{\text{gall.}}{1} = \overset{s.}{9} \\ 20; \textit{ gain} \text{ ,, ,, } = 4 \\ 30; \textit{ loss} \text{ ,, ,, } = 6 \\ 32; \textit{ loss} \text{ ,, ,, } = 8 \end{cases} \qquad \text{Ans.} \begin{cases} \overset{\text{galls.}}{6} \text{ @ } \overset{s.}{15} = \overset{s.}{90} \\ 8 \text{ ,, } 20 = 160 \\ 9 \text{ ,, } 30 = 270 \\ 4 \text{ ,, } 32 = 128 \end{cases}$$

$$27 \text{ ,, } 24 = 648$$

$$24s. \begin{cases} \overset{s.}{15}; \textit{ gain} \text{ on } \overset{\text{gall.}}{1} = \overset{s.}{9} \\ 20; \textit{ gain} \text{ ,, ,, } = 4 \\ 30; \textit{ loss} \text{ ,, ,, } = 6 \\ 32; \textit{ loss} \text{ ,, ,, } = 8 \end{cases} \qquad \text{Ans.} \begin{cases} \overset{\text{galls.}}{8} \text{ @ } \overset{s.}{15} = \overset{s.}{120} \\ 6 \text{ ,, } 20 = 120 \\ 4 \text{ ,, } 30 = 120 \\ 9 \text{ ,, } 32 = 288 \end{cases}$$

$$27 \text{ ,, } 24 = 648$$

EXAMPLE IV.—There are three different kinds of rum—worth 16*s*., 15*s*., and 8*s*. a gallon, respectively; what quantity of each would be required to form 120 gallons of a mixture worth 11*s*. a gallon?

We first find the *proportions* in which the three kinds should be mixed. On every gallon of the—

s. 16 rum, there would be a *loss* of	*s.* 5
15 ,, ,, ,, ,, ,, *loss* ,,	4
8 ,, ,, ,, ,, ,, *gain* ,,	3

In order, therefore, that the loss on the first rum may be counterbalanced by the gain on the third, the mixture should contain 5 gallons of the latter for every 3 of the former; and that the loss on the second rum may be counterbalanced by the gain on the third, there should be 4 gallons of the 8*s*. for every 3 of the 15*s*. kind:

$$\textit{Loss} \text{ on } \begin{cases} \overset{\text{galls.}}{3} \text{ of the } \overset{s.}{16} \text{ rum} = 3 \times 5s. \\ 3 \text{ ,, ,, } 15 \text{ ,, } = 3 \times 4s. \end{cases} \Big\} \text{Total } \textit{loss} = 3 \times (5+4)s.$$

$$\textit{Gain} \text{ ,, } 5+4 \text{ ,, ,, } 8 \text{ ,, } = \qquad 3 \times (5+4)s.$$

$$11s. \begin{cases} \overset{s.}{16}; \textit{ loss} \text{ on } \overset{\text{gall.}}{1} = \overset{s.}{5} \\ 15; \textit{ loss} \text{ ,, ,, } = 4 \\ 8; \textit{ gain} \text{ ,, ,, } = 3 \end{cases} \qquad \text{Proportions:} \begin{cases} \overset{\text{galls.}}{3} \text{ @ } \overset{s.}{16} = \overset{s.}{48} \\ 3 \text{ ,, } 15 = 45 \\ 9 \text{ ,, } 8 = 72 \end{cases}$$

$$15 \text{ ,, } 11 = 165$$

Dividing 120 into three parts proportional to the numbers 3, 3, and 9, we thus find the required quantities to be 24 gallons of the 16*s*., 24 of the 15*s*., and 72 of the 8*s*. rum:

$$3+3+9=15;$$
$$15:3::120:a;\ a=24$$
$$15:3::120:b;\ b=24$$
$$15:9::120:c;\ c=72$$

galls.		*s.*		*s.*
24	@	16	=	384
24	„	15	=	360
72	„	8	=	576
120	„	11	=	1,320

NOTE.—The "proportions" found in the last example are by no means the only ones which would answer. Thus, if 7 gallons (say) of the 16*s*. and 13 gallons of the 15*s*. rum were taken, there would be a loss—on the former, of (7×5=) 35*s*.; on the latter, of (13×4=) 52*s*.; altogether, of (35+52=) 87*s*. Dividing 87*s*., therefore, by 3*s*.—the gain on a gallon of the 8*s*. rum, we see that the necessary quantity of this rum would be (87÷3=) 29 gallons:

alls.		*s.*		*s.*
7	@	16	=	112
13	„	15	=	195
29	„	8	=	232
49	„	11	=	539

So that if 120 were divided into parts proportional to 7, 13, and 29, those parts would fulfil the required conditions.

POWERS AND ROOTS.

214. A number which is the product of two or more *equal* factors is said to be a POWER of one of those factors; and a number from whose repetition, as factor, a certain product can be obtained is called a ROOT of that product.

We say that 49 is a *power* of 7, or that 7 is a *root* of 49; that 125 is a *power* of 5, or that 5 is a *root* of 125; that 1,296 is a *power* of 6, or that 6 is a *root* of 1,296; and so on:

$$49=7\times7;\ 125=5\times5\times5;\ 1{,}296=6\times6\times6\times6;\ \&c.$$

215. The number denoting *how many times* the root is contained, as factor, in the power is termed the INDEX.

We say that 49 is the *second* power of 7, or that 7 is the *second* root of 49; that 125 is the *third* power of 5, or that 5 is the *third* root of 125; that 1,296 is the *fourth* power of 6, or that 6 is the *fourth* root of 1,296; and so on. Here the *indices* are 2, 3, 4, &c., and are written in either of the following two ways:—

$49=7^2$	$7=\sqrt[2]{49}$
$125=5^3$	$5=\sqrt[3]{125}$
$1{,}296=6^4$	$6=\sqrt[4]{1{,}296}$
&c. &c.	&c. &c.

It will be observed that 7^2, 5^3, and 6^4 are short expressions for 7×7, $5\times5\times5$, and $6\times6\times6\times6$, respectively. It will also be observed that "$49=7^2$" and "$7=\sqrt[2]{49}$" are two different statements of the one fact: "$49=7^2$" meaning that 49 is the second *power* of 7; and "$7=\sqrt[2]{49}$," that 7 is the second *root* of 49. Again, "$125=5^3$" and "$5=\sqrt[3]{125}$" are two different statements of the one fact, and the remark is equally applicable to the expressions "$1{,}296=6^4$" and "$6=\sqrt[4]{1{,}296}$": because, to say that 125 is the third *power* of 5 is exactly the same as to say that 5 is the third *root* of 125; and to say that 1,296 is the fourth *power* of 6 is the same as to say that 6 is the fourth *root* of 1,296.

NOTE.—In expressing the second root of a number, we usually omit the index (2). Thus, for the second root of 49 we write $\sqrt{49}$, instead of $\sqrt[2]{49}$.

216. The second power of a number is commonly spoken of as the SQUARE of the number, and the second root as the SQUARE root.

This is explained by the fact that the area of a square is expressed by the second power of the number denoting the length of one of the sides. Thus, if the length of the side were 8 yards, the area of the square would be ($8^2=8\times8=$) 64 square yards.

217. The third power of a number is generally spoken of as the CUBE of the number, and the third root as the CUBE root.

This is accounted for by the fact that the solidity or capacity of a cube is expressed by the third power of the number denoting the length of one of the edges. Thus, if the edge were 4 inches in length, the cube would contain ($4^3=4\times 4\times 4=$) 64 cubic inches.

INVOLUTION.

218. The finding of powers, when roots and indices are given, is called INVOLUTION.

Every exercise in Involution, it is hardly necessary to observe, is worked by multiplication. Thus, to find the fifth power of 7, we simply set down 7, as factor, five times, and then multiply the factors together: $7^5=7\times 7\times 7\times 7\times 7=16{,}807$. Here 7, the root, would be said to have been raised (or "involved") to the fifth power.

219. The product of two or more powers of the *same* number is, itself, a power of that number: a power whose index is the sum of the indices of the powers employed as factors.

Thus, $8^7\times 8^4=(8\times 8\times 8\times 8\times 8\times 8\times 8)\times(8\times 8\times 8\times 8)=8^{11}$ —*i.e.*, 8^{7+4}; $6^5\times 6^3\times 6^2=(6\times 6\times 6\times 6\times 6)\times(6\times 6\times 6)\times(6\times 6)=6^{10}$—*i.e.*, 6^{5+3+2}; &c.

General formula: $x^a\times x^b\times x^c=x^{a+b+c}$.

220. The index is doubled when a power is squared, trebled when a power is cubed, and so on.

This follows from § 219. The square of 5^2, for instance, is $5^2\times 5^2=5^4$; the square of 5^3 is $5^3\times 5^3=5^6$; the square of 5^4 is $5^4\times 5^4=5^8$; &c. In like manner, the cube of 5^2 is $5^2\times 5^2\times 5^2=5^6$; the cube of 5^3 is $5^3\times 5^3\times 5^3=5^9$; the cube of 5^4 is $5^4\times 5^4\times 5^4=5^{12}$; &c.

General formula: nth power of $x^a=x^{a\times n}$.

221. When a power of a number is divided by another power of the *same* number, the resulting quotient is a power of that number: a power whose index is what remains when the index of the power

employed as divisor is subtracted from the index of the power employed as dividend.

This also follows from §219. For instance, $\frac{8^{11}}{8^4}=\frac{8^7\times 8^4}{8^4}=8^7$ —*i.e.*, 8^{11-4}; $\frac{6^{10}}{6^2}=\frac{6^8\times 6^2}{6^2}=6^8$—*i.e.*, 6^{10-2}; &c.

General formula : $\frac{x^a}{x^b}=x^{a-b}$.

222. When UNITY is expressed as a power of *any* number, the index is *nought* (0).

In order to establish this fact, which follows from §221, we have merely to divide any power by itself. Thus, $\frac{5^3}{5^3}=5^{3-3}=5^0$; $\frac{7^4}{7^4}=7^{4-4}=7^0$; $\frac{x^a}{x^a}=x^{a-a}=x^0$; &c. But such fractions as $\frac{5^3}{5^3}$, $\frac{7^4}{7^4}$, $\frac{x^a}{x^a}$, &c. are each equal to 1—the division of a number by itself giving unity for quotient. Consequently, 1 is equal to 5^0, to 7^0, to x^0, &c.

223. When a number is expressed as a power of *itself*, the index is 1.

Thus, $5=5^1$; $7=7^1$; $x=x^1$; &c.

224. When a number is resolvable into two or more factors, we can raise it to any particular power by raising each of the factors to that power, and multiplying those powers together.

Thus, $77^2=(11\times 7)^2=\overline{11\times 7}\times\overline{11\times 7}=\underline{(11\times 11)}\times\underline{(7\times 7)}$ $=11^2\times 7^2$; $105^3=(7\times 5\times 3)^3=\overline{7\times 5\times 3}\times\overline{7\times 5\times 3}\times\overline{7\times 5\times 3}$ $=(7\times 7\times 7)\times(5\times 5\times 5)\times(3\times 3\times 3)=7^3\times 5^3\times 3^3$; &c.

225. When a number ends with one or more ciphers, its square ends with twice as many.

Thus, $130^2=(13\times 10)^2=$ [§224] $13^2\times 10^2=169\times 100=$ 16,900; $3,400^2=(34\times 100)^2=34^2\times 100^2=1,156\times 10,000$ $=11,560,000$; $26,000^2=(26\times 1,000)^2=26^2\times 1,000^2=676\times$ $1,000,000=676,000,000$; &c.

226. When a number ends with one or more ciphers, its cube ends with three times as many.

Thus, $250^3=(25\times10)^3=$ [§224] $25^3\times10^3=15{,}625\times1{,}000$ $=15{,}625{,}000$; $400^3=(4\times100)^3=4^3\times100^3=64\times1{,}000{,}000$ $=64{,}000{,}000$; $7{,}000^3=(7\times1{,}000)^3=7^3\times1{,}000^3=343\times$ $1{,}000{,}000{,}000=343{,}000{,}000{,}000$; &c.

227. The square of the sum of any two numbers exceeds the sum of their squares by twice the product of the numbers.

The square of the sum of 9 and 7, for instance, contains (*a*) the square of 9, (*b*) twice the product of 9 by 7, and (*c*) the square of 7. For, on performing the multiplication in the manner shown below, we see that the first partial product contains 7^2 and 9×7; that the second partial product contains 9^2 and 9×7; and that, consequently, the sum of the two partial products contains 9^2, 7^2, and the double of 9×7:—

$$\begin{array}{rl}
9^2 &=81\\
2\times9\times7 &=126\\
7^2 &=\underline{49}\\
(9+7)^2,\text{ or }16^2 &=256
\end{array}$$

$$\begin{array}{rl}
9+7 & \\
\underline{9+7} & \\
9\times7+7^2 = 7 & \text{times the multiplicand}\\
\underline{9^2+9\times7\quad\;\, =9} & \text{,,}\quad\text{,,}\quad\text{,,}\\
9^2+2\times9\times7+7^2=16 & \text{,,}\quad\text{,,}\quad\text{,,}\quad =16^2,\text{ or }(9+7)^2.
\end{array}$$

General formula: $(x+y)^2=x^2+2xy+y^2$.

228. The cube of the sum of any two numbers consists of four parts, namely: (*a*) the cube of the first number, (*b*) three times the product of the square of the first number by the second, (*c*) three times the product of the first number by the square of the second, and (*d*) the cube of the second number.

Thus, the cube of the sum of 9 and 7 contains (*a*) 9^3, (*b*) $3\times9^2\times7$, (*c*) $3\times9\times7^2$, and (*d*) 7^3. In order to understand this, we have merely to take $9^2+2\times9\times7+7^2$—that is, $(9+7)^2$—for multiplicand, and $9+7$ for multiplier:

$$\begin{array}{rl}
9^3 &=729\\
3\times9^2\times7 &=1{,}701\\
3\times9\times7^2 &=1{,}323\\
7^3 &=\underline{\;\;343}\\
(9+7)^3,\text{ or }16^3 &=4{,}096
\end{array}$$

$$\begin{array}{rl}
9^2+2\times9\times7+7^2 & \\
\underline{9+7} & \\
9^2\times7+2\times9\times7^2+7^3=\;7 & \text{times the multiplicand}\\
\underline{9^3+2\times9^2\times7+9\times7^2\quad\;\, =\;9} & \text{,,}\quad\text{,,}\quad\text{,,}\\
9^3+3\times9^2\times7+3\times9\times7^2+7^3=16 & \text{,,}\quad\text{,,}\quad\text{,,}\quad =16^3,
\end{array}$$
or $(9+7)^3$.

General formula: $(x+y)^3=x^3+3x^2y+3xy^2+y^3$.

SQUARES AND CUBES OF THE FIRST TEN (INTEGRAL) NUMBERS:

Numbers,	1	2	3	4	5	6	7	8	9	10
Squares,	1	4	9	16	25	36	49	64	81	100
Cubes,	1	8	27	64	125	216	343	512	729	1,000

EVOLUTION.

229. The finding of roots, when powers and indices are given, is called EVOLUTION.

We work an exercise in Evolution when, for instance, we ascertain that the square root of 529 is 23, or that the cube root of 4,913 is 17, or that the fourth root of 20,736 is 12; in other words, when we "extract" (or "evolve") the square root of 529, or the cube root of 4,913, or the fourth root of 20,736.

"Square" numbers—that is, numbers which (like 1, 4, 9, 16, 25, &c.) have exact square roots—are comparatively few; and "cube" numbers—that is, numbers which (like 1, 8, 27, 64, 125, &c.) have exact cube roots—are fewer still. So that, in the great majority of instances, what is termed a square root or a cube root is merely an *approximation*. The number 6 for example, has neither an exact square root nor an exact cube root. As 6 lies between 4 and 9, the square root of 6, if determinable, would lie between the square root of 4 and the square root of 9—that is, between 2 and 3. The square root of 6, therefore, if a definite number, would exceed 2 by some proper fraction, and would be convertible into an improper fraction having, *when in its simplest form*, x (say) for numerator, and y for denominator. Now, if $\frac{x}{y}$—a fraction in its simplest form—were the square root of 6, $\frac{x}{y} \times \frac{x}{y}$ or $\frac{x^2}{y^2}$ would be equal to 6; so that y^2 would be a measure of x^2. But this is impossible: because x, being prime to y, must (p. 141) be prime to y^2; and y^2, being prime to x, must be prime to x^2. Consequently, 6 has not an exact square root; and it could be shown, in the same way, that 6 has not an exact cube root.

For the square root of 6, we can write 2, or 2·4, or 2·44, or 2·449, or 2·4494, or 2·44948, or 2·449489,* &c.; the number of approximations—each closer than the preceding one—being unlimited. Here, then, is a repetition of the paradox noticed at pp.

* *This number is the* square root of 5·999996361121, which differs *from 6 by less than* ·000004.

143–4: we can go on "*continually* approaching, without ever reaching," the square root of a number which is not a "square" number, or the cube root of a number which is not a "cube" number.

In the decimal (·449489 &c.) resulting from the extraction of the square root of 6, we have an illustration of a class of decimals which neither terminate nor circulate. The decimal referred to would, if terminate, be convertible into a decimal fraction (§ 141); and would, if a circulating decimal, be convertible into a vulgar fraction (§§ 153, 154). But we have just seen that there is *no* fraction which, when added to 2, would give the square root of 6; so that the decimal ·449489 &c. is interminate and non-circulating.

EXTRACTION OF THE SQUARE ROOT.

230. When the integral part of a number occupies either—

$$\left.\begin{array}{l} 1 \text{ or } 2 \text{ places,} \\ 3 \text{ ,, } 4 \text{ ,,} \\ 5 \text{ ,, } 6 \text{ ,,} \\ 7 \text{ ,, } 8 \text{ ,,} \\ \quad\&\text{c.} \end{array}\right\} \begin{array}{c} \text{the integral part of} \\ \text{the square root of the} \\ \text{number will occupy} \end{array} \left\{\begin{array}{l} 1 \text{ place} \\ 2 \text{ places} \\ 3 \text{ ,,} \\ 4 \text{ ,,} \\ \&\text{c.} \end{array}\right.$$

Thus, as every number whose integral part is expressed by either one figure or two must be less than 100, but not less than 1, the square root of every such number must be less than $\sqrt{100}$, but not less than $\sqrt{1}$—that is, must be less than 10, but not less than 1; so that the integral part of the square root will occupy one place. Again: as every number whose integral part is expressed by either three or four figures must be less than 10,000, but not less than 100, the square root of every such number must be less than $\sqrt{10,000}$, but not less than $\sqrt{100}$—that is, must be less than 100, but not less than 10; so that the integral part of the square root will occupy two places. In like manner, as every number whose integral part is expressed by either five or six figures must be less than 1,000,000, but not less than 10,000, the square root of every such number must be less than $\sqrt{1,000,000}$, but not less than $\sqrt{10,000}$—that is, must be less than 1,000, but not less than 100; so that the integral part of the square root will occupy three places. We see, therefore, *without proceeding* further, that if the integral part of a

number occupied n places, the integral part of the square root of the number would occupy either $\frac{n}{2}$ or $\frac{n+1}{2}$ places—according as n happened to be even or odd.

EXAMPLE I.—Extract $\sqrt{1,444}$.

This number being expressed by four figures, its square root will (§ 230) be expressed by two—a tens' and a units' figure. In determining how many tens there are in the root, we disregard the last two figures (44) of 1,444—knowing that (§ 225) the square of any number of tens can contain nothing lower than *hundreds*. As 14 lies between 9 and 16, or between 3^2 and 4^2, 1,400—and therefore 1,444—must lie between 30^2 and 40^2; so that the tens' figure of the root is 3.

Putting u for the number of units in the root, we have $\sqrt{1,444} = 30+u$; or $1,444 = (30+u)^2 = 30^2+2\times 30\times u+u^2$. When, therefore, 30^2, or 900, is subtracted from 1,444, there remains $544=2\times 30\times u+u^2=60\times u+u^2$. As $60\times u$ is evidently a much larger number than u^2, it follows that $60\times u$ constitutes the principal portion of 544, and that u must be *nearly* equal to the 60th part of 544, or to the 6th part of 54—that is, to 9. Now, if 9 be the units' figure of the root, we shall have $544 = 60\times 9+9^2=(60+9)\times 9=69\times 9$. But 69×9, or 621, exceeds 544; so that the units' figure of the root must be a lower one than 9. We therefore try 8; and, finding $544=60\times 8+8^2= (60+8)\times 8=68\times 8$, we conclude that the units' figure *is* 8, and that the required root is $(30+8=)38$.

EXAMPLE II.—Extract $\sqrt{208,849}$.

This number being expressed by six figures, its square root will (§ 230) be expressed by three—a hundreds', a tens', and a units' figure. Remembering that (§ 225) the square of any number of hundreds terminates with four ciphers, we disregard the last four figures (8,849) of 208,849, and consider only the left-hand pair (20), whilst determining the hundreds' figure of the root. As 20 lies between 16 and 25, or between 4^2 and 5^2, 200,000—and therefore 208,849—must lie between 400^2 and 500^2; so that the hundreds' figure of the root is 4. Putting x for the undiscovered part of the root, we have $\sqrt{208,849}=400+x$; or $208,849=(400+x)^2=400^2+2\times 400\times x+x^2$. When, therefore, 400^2, or 160,000, is subtracted from 208,849, there remains $48,849=2\times 400\times x+x^2=800\times x+x^2$. As $800\times x$—being a much larger number than x^2—constitutes the principal portion of 48,849, we divide 48,849 by 800, or 488 by 8, and find that, the undiscovered part of the root being less than 70, the tens' figure cannot be higher than 6. Adding 60^2 to 800×60,—or multiplying $(800+60=)860$ by 60,—we obtain 51,600, which exceeds 48,849; so that the

tens' figure of the root must be lower than 6. We therefore try 5. Adding 50^2 to 800×50,—or multiplying ($800+50=$) 850 by 50,—we find that the result, 42,500, is contained in 48,849; for which reason we conclude that the tens' figure of the root is 5. The "known" part of the root is now ($400+50=$)450. Putting y for the part still undiscovered, we have $\sqrt{208,849}=450+y$; or $208,849=(450+y)^2=450^2+2\times 450\times y+y^2$. When, therefore, 450^2 is subtracted from 208,849, the remainder, 6,349, is equal to $2\times 450\times y+y^2=900\times y+y^2$.

[This remainder is most easily obtained when we subtract 42,500 from 48,849 : $450^2=(400+50)^2=400^2+2\times 400\times 50+50^2=160,000+800\times 50+50^2=160,000+850\times 50=160,000+42,500$; $208,849-160,000=48,849$; $48,849-42,500=6,349$.]

As $900\times y$—being very much larger than y^2—constitutes the principal portion of 6,349, the division of 6,349 by 900, or of 63 by 9, will probably give the remaining part of the root. The quotient being 7, we add 7^2 to 900×7, or multiply ($900+7=$) 907 by 7; and, finding the result to be exactly 6,349, we conclude that the units' figure of the root is 7, and that the root is 457.

Example III.—Extract $\sqrt{864,735,219}$.

This number being expressed by nine figures, the integral part of its square root will (§ 230) be expressed by five $\left(\frac{9+1}{2}\right)$; so that the most left-hand figure of the root will represent one or more groups of 10,000 each. As the square of any number of such groups (§ 225) terminates with eight ciphers, we disregard the last eight figures (64735219) of 864,735,219, and consider only the most left-hand one (8), whilst determining the most left-hand figure of the root. Seeing that 8 lies between 4 and 9, or between 2^2 and 3^2, we know that 800,000,000—and therefore 864,735,219—must lie between $20,000^2$ and $30,000^2$. The most left-hand figure of the root, therefore, is 2. Putting x for the remainder of the root, we have $\sqrt{864,735,219}=20,000+x$; or $864,735,219=(20,000+x)^2=20,000^2+2\times 20,000\times x+x^2$. When, therefore, $20,000^2$, or 400,000,000, is subtracted from 864,735,219, there remains $464,735,219=2\times 20,000\times x+x^2=40,000\times x+x^2$. As $40,000\times x$—being a much larger number than x^2—constitutes the principal portion of 464,735,219, the division of 464,735,219 by 40,000, or of 46,473 by 4, enables us to approximate to the undiscovered portion of the root. Although the quotient exceeds 9,000, (4 being contained more than 9 times in 46,) we know that the next figure of the root cannot be higher than 9, which, therefore, we try. Adding $9,000^2$ to $40,000\times 9,000$,—or multiplying

$(40,000+9,000=)$ $49,000$ by $9,000$,—we find that the result, $441,000,000$, is contained in $464,735,219$; so that the next figure of the root must be 9. The "known" part of the root is now $(20,000+9,000=)$ $29,000$. Putting y for the "unknown" part, we have $\sqrt{864,735,219}=29,000+y$; or $864,735,219=(29,000+y)^2=29,000^2+2\times 29,000\times y+y^2$. When, therefore, $29,000^2$ is subtracted from $864,735,219$, the remainder, $23,735,219$, is equal to $2\times 29,000\times y+y^2=58,000\times y+y^2$.

[This remainder is most easily obtained when we subtract $441,000,000$ from $464,735,219$: $29,000^2=(20,000+9,000)^2=20,000^2+2\times 20,000\times 9,000+9,000^2=400,000,000+40,000\times 9,000+9,000^2=400,000,000+49,000\times 9,000=400,000,000+441,000,000$; $864,735,219-400,000,000=464,735,219$; $464,735,219-441,000,000=23,735,219$.]

As $58,000\times y$—being much larger than y^2—constitutes the principal portion of $23,735,219$, the division of $23,735,219$ by $58,000$, or of $23,735$ by 58, enables us to approximate to the remaining part of the root. The quotient being more than 400, and less than 500, the hundreds' figure of the root cannot be higher than 4. Adding 400^2 to $58,000\times 400$, or multiplying $(58,000+400=)$ $58,400$ by 400, we find that the result, $23,360,000$, is contained in $23,735,219$; so that the hundreds' figure of the root must be 4. The "known" part of the root is now $(29,000+400=)$ $29,400$. Putting z for the "unknown" part, we have $\sqrt{864,735,219}=29,400+z$; or $864,735,219=(29,400+z)^2=29,400^2+2\times 29,400\times z+z^2$. When, therefore, $29,400^2$ is subtracted from $864,735,219$, the remainder, $375,219$, is equal to $2\times 29,400\times z+z^2=58,800\times z+z^2$.

[This remainder is most easily obtained when we subtract $23,360,000$ from $23,735,219$: $29,400^2=(29,000+400)^2=29,000^2+2\times 29,000\times 400+400^2=29,000^2+58,000\times 400+400^2=29,000^2+(58,000+400)\times 400=29,000^2+58,400\times 400=29,000^2+23,360,000$; $864,735,219-29,000^2=23,735,219$; $23,735,219-23,360,000=375,219$.]

As $58,800\times z$—being very much larger than z^2—constitutes the principal portion of $375,219$, the division of $375,219$ by $58,800$, or of $3,752$ by 588, enables us to approximate to the remainder of the root. The quotient being less than 10, the tens' figure of the root must be 0, which we accordingly set down. We next look for the units' figure, which cannot be higher than 6; the divisor being contained 6 times, but not 7 times, in the dividend. Adding 6^2 to $58,800\times 6$,—or multiplying $(58,800+6=)$ $58,806$ by 6,—we find that the result, $352,836$, is contained in $375,219$; so that the units' figure of the root is 6.

Subtracting $352,836$ from $375,219$, we see that the given number $(864,735,219)$ exceeds the square of $29,406$ by $22,383$;

$29{,}406^2 = (29{,}400+6)^2 = 29{,}400^2 + 2 \times 29{,}400 \times 6 + 6^2 = 29{,}400^2 + 58{,}800 \times 6 + 6^2 = 29{,}400^2 + 58{,}806 \times 6 = 29{,}400^2 + 352{,}836$; $864{,}735{,}219 - 29{,}400^2 = 375{,}219$; $375{,}219 - 352{,}836 = 22{,}383$. The square root of 864,735,219, therefore, lies between 29,406 and 29,407.

Here is the work, both in full and in a contracted form, of the preceding examples:—

Example I.

Uncontracted Process.

$$
\begin{array}{rr|l}
 & 1444 & 30 \\
 & 900 & +8 \\ \cline{2-2}
 & 544 & \\
\left.\begin{array}{r} 60 \\ +8 \end{array}\right\} \times 8 = & 544 & \\ \cline{2-2}
\end{array}
$$

Contracted Process.

$$
\begin{array}{rr|l}
 & 1\dot{4}4\dot{4} & 38 \\
 & 9 & \\ \cline{2-2}
 & 544 & \\
68 & 544 & \\ \cline{2-2}
\end{array}
$$

Example II.

Uncontracted Process.

$$
\begin{array}{rr|l}
 & 208849 & 400 \\
 & 160000 & +50 \\ \cline{2-2}
 & & +7 \\
 & 48849 & \\
\left.\begin{array}{r} 800 \\ +50 \end{array}\right\} \times 50 = & 42500 & \\ \cline{2-2}
 & 6349 & \\
\left.\begin{array}{r} 900 \\ +7 \end{array}\right\} \times 7 = & 6349 & \\ \cline{2-2}
\end{array}
$$

Contracted Process.

$$
\begin{array}{rr|l}
 & \dot{2}08\dot{8}4\dot{9} & 457 \\
 & 16 & \\ \cline{2-2}
 & 488 & \\
85 & 425 & \\ \cline{2-2}
 & 6349 & \\
907 & 6349 & \\ \cline{2-2}
\end{array}
$$

Example III.

Uncontracted Process.

$$
\begin{array}{rr|r}
 & 864735219 & 20{,}000 \\
 & 400000000 & +9{,}000 \\ \cline{2-2}
 & & +400 \\
 & 464735219 & +6 \\
\left.\begin{array}{r} 40{,}000 \\ +9{,}000 \end{array}\right\} \times 9{,}000 = & 441000000 & \\ \cline{2-2}
 & 23735219 & \\
\left.\begin{array}{r} 58{,}000 \\ +400 \end{array}\right\} \times 400 = & 23360000 & \\ \cline{2-2}
 & 375219 & \\
\left.\begin{array}{r} 58{,}800 \\ +6 \end{array}\right\} \times 6 = & 352836 & \\ \cline{2-2}
 & 22383 & \\
\end{array}
$$

Contracted Process.

$$
\begin{array}{rr|l}
 & \dot{8}6\dot{4}7\dot{3}5\dot{2}1\dot{9} & 29406 \\
 & 4 & \\ \cline{2-2}
 & 464 & \\
49 & 441 & \\ \cline{2-2}
 & 2373 & \\
584 & 2336 & \\ \cline{2-2}
 & 375219 & \\
58806 & 352836 & \\ \cline{2-2}
 & 22383 & \\
\end{array}
$$

It thus appears that the square root of a number is extracted figure by figure; that, after the finding of the first or most left-hand figure, the root, whilst being extracted, is regarded as consisting of two parts—the "known" and the "unknown"; that, as the work progresses, the known part of the root becomes gradually larger, and the unknown part gradually smaller; that, every time a new root-figure is obtained, the given number is diminished by the square of the part of the root then known;* that the resulting remainder consists principally of the product of twice the known by the unknown part of the root; and that, consequently, the division of the remainder by twice the known part must give the next figure of the root, or a figure not much higher—according as the remainder does or does not contain, in addition to the product in question, the square of the quotient.

In the contracted process, we simply dispense with unnecessary ciphers, and (instead of writing the remainders in full) "bring down" the figures according as they are required—that is, in pairs: the last pair being made no use of until the tens' figure of the root has been determined; the last two pairs being made no use of until the hundreds' figure of the root has been determined; and so on, as we proceed to the left. (See § 225.) When, therefore, we place a dot over every alternate figure—beginning with the units' figure—of a number whose square root is to be extracted, we see at a glance what particular pair of figures—or what particular "period"—is to be brought down at each stage of the work; and, by counting the dots, we are able to tell, beforehand, how many places the root will occupy—a place for every dot.

231. To extract the Square Root of a (whole) number: Divide the figures into periods of two each, by placing a dot over every alternate one, beginning with the units' figure; and should there be an odd figure remaining, on the extreme left, regard it also as a period. Find the highest figure whose square is contained in the most left-hand period, and set it down as the most left-hand

---

* From an examination of the last example it will be seen that 864,735,219 exceeds the square of the first root-figure, or of 20,000, by 464,735,219; the square of the first two root-figures, or of 29,000, by 23,735,219; the square of the first three (and therefore of the first four) root-figures, or of 29,400, by 375,219; and the square of the whole five root-figures, or of 29,406, by 22,383.

figure of the required root. Subtract the square of this figure from the first period, and to the remainder annex the second period: the number thus obtained will be the first "dividend," which, when the last figure is conceived to be cut off, will be the first "trial"-dividend. As the first "trial-divisor," set down the double of the root-figure. Divide the trial-dividend by the trial-divisor, and annex the resulting quotient-figure to the root-figure, and also to the trial-divisor. Multiply the trial-divisor, in its thus altered form, by the quotient-figure; and if the product be not contained in the dividend, substitute for the quotient-figure a lower one—both after the root-figure and after the trial-divisor. Having found the highest figure which, employed in the manner just described, gives a product that does not exceed the dividend, set it down as the second root-figure, and subtract the product from the dividend. To the remainder annex the third period, and the resulting number will be the second dividend, which, when the last figure is conceived to be cut off, will be the second trial-dividend. As the second trial-divisor, set down the double of the part of the root now known. With the new trial-dividend and trial-divisor proceed as before; and continue the process until the last period shall have been brought down. The required root will then have been obtained if, after the bringing down of the last period, there be no remainder; but should there be a remainder, the number obtained for root will be the square root—not of the given number, but—of the difference between the given number and this remainder.

Thus, in the case of the last example (contracted process), there are five dots, which divide the figures of the given number into five periods—the odd figure (8) on the extreme left being regarded as a period. The square root of the largest square contained in 8 being 2, we set down 2 as the first figure of the root. Subtracting the square of this figure from 8, and bring-

ing down the next period (64), we obtain the first "dividend," 464. The first "divisor," 49, we obtain by annexing 9 to the double of the first figure (2) of the root: 9 being first conjectured, and afterwards ascertained, to be the second figure of the root. The division of 464 by 49 being pretty much the same as the division of 46 by 4, we employ 46 and 4 as "trial"-dividend and "trial"-divisor, respectively, in our search for the root-figure 9. Subtracting 49×9 from 464, and bringing down the next period (73) to the remainder (23), we obtain the second dividend, 2373. The second divisor, 584, we obtain by annexing 4 to the double of the known part (29) of the root: 4 being first conjectured, and afterwards ascertained, to be the third figure of the root. The division of 2373 by 584 being pretty much the same as the division of 237 by 58, we employ 237 and 58 as trial-dividend and trial-divisor, respectively, in our search for 4. Subtracting 584×4 from 2373, and bringing down the next period (52) to the remainder (37), we obtain the third dividend, 3752. Conceiving the last figure (2) of this number cut off, we have for trial-dividend 375; and, doubling 294, the known part of the root, we obtain 588 for trial-divisor. As 588 is not contained in 375, we write 0 as the next figure of the root: we also annex a cipher to 588, in order to double 2940, the part of the root now known. Bringing down the last period (19), we have 375219 for dividend, and 37521 for trial-dividend. Employing 5880 for trial-divisor, we obtain 6 for quotient. Annexing 6 to 5880, and finding that 58806 is contained 6 times in 375219, we write 6 as the units' figure of the root. Lastly, subtracting 58806×6 from 375219, we find that the given number (864,735,219) exceeds the square of 29,406 by 22,383.

NOTE 1.—When the division of a trial-dividend by a trial-divisor gives 0 for quotient-figure, or when 1 is obtained for quotient-figure, and is afterwards found to be too high, write a cipher after the known part of the root, and also after the trial-divisor; bring down the next period; and treat the resulting numbers as a new trial-divisor and trial-dividend, respectively.

NOTE 2.—After having, in the ordinary way, found *more than* half the figures of a square root, we can obtain the others by division only, as follows: For divisor, double the known part of the root; for dividend, bring down the outstanding periods—cutting off (from the right-hand side) a figure for every undiscovered root-figure; the resulting quotient—the remainder being disregarded—will be the required part of the root.

Thus, in extracting the square-root of 11,949,390,190,521, we need not work, in the ordinary way, for more than the first *four* figures (3456)—knowing that the total number is *seven*.

Doubling 3456 (thousands), we obtain 6912 (thousands), which we employ as divisor; bringing down the outstanding periods, and cutting off the last three figures,—that is, as many as there are root-figures still to be found,—we obtain 5454190 (thousands), which we employ as dividend; and in the resulting quotient (789)—the remainder being disregarded—we have the last three figures of the root:

	11949390190521	3456
	9	
64	294	
	256	
685	3893	
	3425	
6906	46890	
	41436	
	5454	

$$3456\times2=6912;\ 5454190\div6912=789;$$
$$\text{required root}=3456789.$$

This is easily explained. Putting x for the last three figures of the root, we have $\sqrt{11949390190521}=3456000+x$; $11949390190521=(3456000+x)^2=3456000^2+2\times3456000\times x+x^2$; $11949390190521-3456000^2=2\times3456000\times x+x^2$; *i.e.*, 5454190521* $=6912000\times x+x^2$; $5454190521\div6912000=x+\frac{x^2}{6912000}$. As the number represented by x occupies only *three* places, the number represented by x^2 cannot occupy more than *six* places (§ 230); so that $\frac{x^2}{6912000}$ is a proper fraction, the denominator occupying *seven* places.† When, therefore, we divide 5454190521 by 6912000,—or 5454190 by 6912,—and reject the remainder, we obtain x.

* From what has already been explained, it is evident that 5454190521 is the difference between 11949390190521 and 3456000^2.

† It is obvious that every such fraction must be proper: the denominator terminating with as many ciphers as there are figures in the square root of the numerator, and the ciphers being preceded by *more than* an equal number of other figures—obtained from the doubling of the known part of the required square root.

232. As the square of a fraction is obtained when the square of the numerator is divided by the square of the denominator—so, the square root of a fraction is obtained when the square root of the numerator is divided by the square root of the denominator.

Thus, $\left(\frac{5}{8}\right)^2=\frac{5}{8}\times\frac{5}{8}=\frac{5\times5}{8\times8}=\frac{5^2}{8^2}=\frac{25}{64}$; $\left(\frac{7}{11}\right)^2=\frac{7}{11}\times\frac{7}{11}=\frac{7\times7}{11\times11}=\frac{7^2}{11^2}=\frac{49}{121}$; &c.: $\sqrt{\frac{25}{64}}=\frac{\sqrt{25}}{\sqrt{64}}=\frac{5}{8}$; $\sqrt{\frac{49}{121}}=\frac{\sqrt{49}}{\sqrt{121}}=\frac{7}{11}$; &c.

233. The following is a comparatively easy way of extracting the square root of a fraction whose terms are not square numbers, or the square roots of whose terms cannot be found by inspection: Multiply the numerator by the denominator, extract the square root of the product, and divide this root by the denominator.

Thus, $\sqrt{\frac{5}{8}}=\sqrt{\frac{5\times8}{8\times8}}=\frac{\sqrt{40}}{\sqrt{8^2}}=\frac{\sqrt{40}}{8}$; $\sqrt{\frac{7}{11}}=\sqrt{\frac{7\times11}{11\times11}}=\frac{\sqrt{77}}{\sqrt{11^2}}=\frac{\sqrt{77}}{11}$; &c.

In the majority of cases, however, the square root of a fraction is most easily extracted when we begin by converting the fraction into a decimal (§ 234).

234. To extract the Square Root of a number which contains a Decimal: Regard the number as an integer—having, if necessary, first made the number of decimal places *even*, by annexing a cipher; and remove the decimal point in the resulting root a place to the left for every *pair* of decimal places in the given number.

As an illustration, let it be required to extract the square root of ·4. The square root of 4 being 2, the square root of ·4 might, at first sight, appear to be ·2. If, however, ·2 were multiplied by itself, the product would be (not ·4, but) ·04. The given decimal can be written under any of the following

forms: $\frac{4}{10}, \frac{40}{100}, \frac{400}{1,000}, \frac{4,000}{10,000}, \frac{40,000}{100,000}, \frac{400,000}{1,000,000}$, &c. Rejecting the fractions ($\frac{4}{10}, \frac{400}{1,000}, \frac{40,000}{100,000}$, &c.) whose denominators are not square numbers, we thus have—

$$\sqrt{\cdot 4} = \frac{\sqrt{40}}{\sqrt{100}} = \frac{6}{10} = \cdot 6 \text{ nearly}$$

$$\sqrt{\cdot 4} = \frac{\sqrt{4,000}}{\sqrt{10,000}} = \frac{63}{100} = \cdot 63 \text{ more nearly}$$

$$\sqrt{\cdot 4} = \frac{\sqrt{400,000}}{\sqrt{1,000,000}} = \frac{632}{1,000} = \cdot 632 \text{ still more nearly}$$

&c. &c.

In practice, ·4 would simply be written ·40, or ·4000, or ·400000, &c., and treated as a whole number—the first figure of the root, however, being made to occupy the first decimal place.

As a second illustration, let it be required to extract the square root of 1234·56789. This number is equivalent to the fraction $\frac{123456789}{100,000}$. As 100,000 is not a square number, we multiply the terms of the fraction by 10: we then have $\sqrt{1234 \cdot 56789} = \frac{\sqrt{1234567890}}{\sqrt{1,000,000}} = \frac{35136}{1,000} = 35 \cdot 136$—a result more easily obtained when we annex a cipher to 1234·56789 (in order to make the number of decimal places even), extract the square root of 1234567890, and remove the decimal point in the root (35136) three places to the left; that is, a place for every pair of figures in ·567890.

Note.—For every figure which we desire to have in the decimal part of the square root of a number, there must be a *pair* of figures—including, if necessary, one or more ciphers—in the decimal part of the number itself. This is rendered obvious by what has just been explained, as well as by the fact that (§ 59) the square of a decimal occupies *twice* as many places as the decimal itself.

235. To extract the Square Root of either a Fractional or a Mixed number: Convert the Fraction into a Decimal, and then proceed as already directed (§ 234).

Thus, in extracting the square root of $\frac{5}{13}$, we write $\frac{5}{13}$ under the form ·384615 &c.; in extracting the square root of $246\frac{3}{8}$, we write $246\frac{3}{8}$ under the form 246·375; and so on.

236. To find an Approximation which shall differ from the Square Root of a number by less than a given Fractional Unit: Multiply the number by the square of the denominator of the fractional unit, and divide the denominator into the square root of the product.

Thus, to find an approximation which shall differ from $\sqrt{47}$ by less than $\frac{1}{8}$, we multiply 47 by 8^2, and divide 8 into the square root of the product: $47=\frac{47\times8^2}{8^2}=\frac{47\times64}{8^2}=\frac{3008}{8^2}$; $\sqrt{47}=\frac{\sqrt{3008}}{8}$. As $\sqrt{3008}$ lies between $\sqrt{2916}$ and $\sqrt{3025}$, or between 54 and 55, $\sqrt{47}$ must lie between $\frac{54}{8}$ and $\frac{55}{8}$; and as these two fractions differ by exactly $\frac{1}{8}$, each differs from $\sqrt{47}$ by less than $\frac{1}{8}$.

Again: to find an approximation which shall differ from $\sqrt{365}$ by less than $\frac{1}{1,000}$, we multiply 365 by $1,000^2$, and divide 1,000 into the square root of the product: $365=\frac{365\times1,000^2}{1,000^2}=\frac{365,000,000}{1,000^2}$; $\sqrt{365}=\frac{\sqrt{365,000,000}}{1,000}$. As $\sqrt{365,000,000}$ lies between $\sqrt{364,962,816}$ and $\sqrt{365,001,025}$, or between 19,104 and 19,105, $\sqrt{365}$ must lie between $\frac{19,104}{1,000}$ and $\frac{19,105}{1,000}$; and as these two fractions differ by exactly $\frac{1}{1,000}$, $\sqrt{365}$ does not differ from either 19·104 or 19·105 by so much as $\frac{1}{1,000}$ (·001).

EXTRACTION OF THE CUBE ROOT.

237. When the integral part of a number occupies—

1, 2, or 3 places,	the integral part of the cube root of the number will occupy	1 place
4, 5, „ 6 „		2 place
7, 8, „ 9 „		3 „
10, 11, „ 12 „		4 „
&c.		&c.

Thus, as every number whose integral part is expressed by

one, two, or three figures must be less than 1,000, but not less than 1, the cube root of every such number must be less than $\sqrt[3]{1,000}$, but not less than $\sqrt[3]{1}$—that is, must be less than 10, but not less than 1; so that the integral part of the root will occupy one place. Again: as every number whose integral part is expressed by four, five, or six figures must be less than 1,000,000, but not less than 1,000, the cube root of every such number must be less than $\sqrt[3]{1,000,000}$, but not less than $\sqrt[3]{1,000}$—that is, must be less than 100, but not less than 10; so that the integral part of the root will occupy two places. In like manner, as every number whose integral part occupies seven, eight, or nine places must be less than 1,000,000,000, but not less than 1,000,000, the cube root of every such number must be less than $\sqrt[3]{1,000,000,000}$, but not less than $\sqrt[3]{1,000,000}$—that is, must be less than 1,000, but not less than 100; so that the integral part of the root will occupy three places. Speaking generally, therefore, we are able to say that when the integral part of a number occupies n places, the integral part of the cube root of the number will occupy $\frac{n}{3}$, or $\frac{n+1}{3}$, or $\frac{n+2}{3}$ places—according as n, or $n+1$, or $n+2$ is a multiple of 3.

EXAMPLE I.—Extract $\sqrt[3]{185,193}$.

This number being expressed by six figures, its cube root must (§ 237) be expressed by two—a tens' and a units' figure. In looking for the tens' figure of the root, we disregard the last three figures (193) of the given number—knowing that (§ 226) the cube of any number of tens is some number of thousands. As 185 lies between 125 and 216, or between 5^3 and 6^3, 185,000—and therefore 185,193—must lie between 50^3 and 60^3; so that the tens' figure of the root is 5. Putting u for the units' figure of the root, we have $\sqrt[3]{185,193}=50+u$; $185,193=(50+u)^3=50^3+3\times50^2\times u+3\times50\times u^2+u^3$; $185,193-50^3=3\times50^2\times u+3\times50\times u^2+u^3$; *i.e.*, $60,193=3\times50^2\times u+3\times50\times u^2+u^3$. As $3\times50^2\times u$ is obviously a much larger number than $3\times50\times u^2+u^3$, the division of 60,193 by 3×50^2, or of 60,193 by 7,500, or of 601 by 75, will give the units' figure of the root, or a figure not much higher. Seeing that 75 is contained 8 times, but not 9 times, in 601, we conclude that the units' figure of the root cannot be higher than 8, which, accordingly, we try:—

$$\begin{array}{r}
3\times50^2\times8=60000\\
3\times50\times8^2=9600\\
8^3=512\\
\hline
70112
\end{array}$$

Or thus:

$$\begin{array}{rl} 158 & [=3\times50+8] \\ 8 & \\ \hline 1264 & [=3\times50\times8+8^2] \\ 7500 & [=3\times50^2] \\ \hline 8764 & [=3\times50^2+3\times50\times8+8^2] \\ 8 & \\ \hline 70112 & [=3\times50^2\times8+3\times50\times8^2+8^3] \end{array}$$

As 70,112 is not contained in 60,193, the units' figure of the root must be lower than 8. We therefore try 7:—

$$\begin{array}{rl} 157 & [=3\times50+7] \\ 7 & \\ \hline 1099 & [=3\times50\times7+7^2] \\ 7500 & [=3\times50^2] \\ \hline 8599 & [=3\times50^2+3\times50\times7+7^2] \\ 7 & \\ \hline 60193 & [=3\times50^2\times7+3\times50\times7^2+7^3] \end{array}$$

Its units' figure being thus found to be 7, the required root is (50+7=) 57.

EXAMPLE II.—Extract $\sqrt[3]{77,308,776}$.

This number being expressed by eight figures, its cube root must (§ 237) be expressed by three—a hundreds', a tens', and a units' figure. In looking for the hundreds' figure of the root, we disregard the last six figures (308,776) of the given number—knowing that (§ 226) the cube of any number of hundreds must be some number of millions. As 77 lies between 64 and 125, or between 4^3 and 5^3, 77,000,000—and therefore 77,308,776—must lie between 400^3 and 500^3; so that the hundreds' figure of the root is 4. Putting x for the remainder of the root, we have $\sqrt[3]{77,308,776}=400+x$; $77,308,776=(400+x)^3=400^3+3\times400^2\times x+3\times400\times x^2+x^3$; $77,308,776-400^3=3\times400^2\times x+3\times400\times x^2+x^3$; *i.e.*, $13,308,776=3\times400^2\times x+3\times400\times x^2+x^3$. As $3\times400^2\times x$ is a much larger number than $3\times400\times x^2+x^3$, the division of 13,308,776 by 3×400^2, or of 13,308,776 by 480,000, or of 1,330 by 48, will give a quotient not much greater than the remainder of the root. The quotient being

more than 20, and less than 30, the tens' figure of the root cannot be higher than 2, which we try :—

$$\begin{array}{rl}
1220 & [=3\times 400+20] \\
20 & \\
\hline
24400 & [=3\times 400\times 20+20^2] \\
480000 & [=3\times 400^2] \\
\hline
504400 & [=3\times 400^2+3\times 400\times 20+20^2] \\
20 & \\
\hline
10088000 & [=3\times 400^2\times 20+3\times 400\times 20^2+20^3]
\end{array}$$

Seeing that 10,088,000 is contained in 13,308,776, we write 2 as the tens' figure of the root. The known part of the root is now $(400+20=)420$. Putting y for the part still undiscovered, we have $\sqrt[3]{77,308,776}=420+y$; $77,308,776=(420+y)^3=420^3+3\times 420^2\times y+3\times 420\times y^2+y^3$. When, therefore, 420^3 is subtracted from 77,308,776, there remains $3,220,776=3\times 420^2\times y+3\times 420\times y^2+y^3$.

[This remainder is most easily obtained when we subtract 10,088,000 from 13,308,776: $420^3=(400+20)^3=400^3+3\times 400^2\times 20+3\times 400\times 20^2+20^3=400^3+10,088,000$; $77,308,776-400^3=13,308,776$; $13,308,776-10,088,000=3,220,776$.]

As $3\times 420^2\times y$ is much larger than $3\times 420\times y^2+y^3$, the division of 3,220,776 by 3×420^2, or of 3,220,776 by 529,200, or of 32207 by 5292, will give either the units' figure of the root or a figure not much higher. The quotient-figure is 6, which we try :—

$$\begin{array}{rl}
1266 & [=3\times 420+6] \\
6 & \\
\hline
7596 & [=3\times 420\times 6+6^2] \\
529200 & [=3\times 420^2] \\
\hline
536796 & [=3\times 420^2+3\times 420\times 6+6^2] \\
6 & \\
\hline
3220776 & [=3\times 420^2\times 6+3\times 420\times 6^2+6^3]
\end{array}$$

Its units' figure being thus found to be 6, the required root is 426.

EXAMPLE III.—Extract $\sqrt[3]{15,783,426,589,234}$.

This number being expressed by fourteen figures, its cube root must (§ 237) be expressed by five $\left(\frac{14+1}{3}\right)$; so that the most left-hand figure of the root will represent a group of 10,000,

or two or more such groups. Knowing that (§226) the cube of any number of groups of 10,000 each is some number of trillions, we confine ourselves, whilst determining the first root-figure, to the 15 trillions which the given number contains. As 15 lies between 8 and 27, or between 2^3 and 3^3, 15,000,000,000,000 —and therefore 15,783,426,589,234—must lie between $20{,}000^3$ and $30{,}000^3$. The first root-figure is thus found to be 2. Putting x for the remainder of the root, we have $\sqrt[3]{15{,}783{,}426{,}589{,}234} = 20{,}000+x$; or $15{,}783{,}426{,}589{,}234=(20{,}000+x)^3=20{,}000^3 + 3\times 20{,}000^2\times x+3\times 20{,}000\times x^2+x^3$. Subtracting $20{,}000^3$, therefore, or 8,000,000,000,000, from 15,783,426,589,234, we obtain $7{,}783{,}426{,}589{,}234=3\times 20{,}000^2\times x+3\times 20{,}000\times x^2+x^3$. As $3\times 20{,}000^2\times x$ is much larger than $3\times 20{,}000\times x^2+x^3$, the division of 7,783,426,589,234 by $3\times 20{,}000^2$, or of 7,783,426,589,234 by 1,200,000,000, or of 77,834 by 12, will give a quotient not much greater than x. The quotient being more than 6,000, and less than 7,000, the second root-figure cannot be higher than 6, which we try:—

66000	$[=3\times 20{,}000+6{,}000]$
6000	
396000000	$[=3\times 20{,}000\times 6{,}000+6{,}000^2]$
1200000000	$[=3\times 20{,}000^2]$
1596000000	$[=3\times 20{,}000^2+3\times 20{,}000\times 6{,}000+6{,}000^2]$
6000	
9576000000000	$[=3\times 20{,}000^2\times 6{,}000+3\times 20{,}000\times 6{,}000^2+6{,}000^3]$

As 9,576,000,000,000 is not contained in 7,783,426,589,234, the second root-figure must be a lower one than 6. We therefore try 5:—

65000	$[=3\times 20{,}000+5{,}000]$
5000	
325000000	$[=3\times 20{,}000\times 5{,}000+5{,}000^2]$
1200000000	$[=3\times 20{,}000^2]$
1525000000	$[=3\times 20{,}000^2+3\times 20{,}000\times 5{,}000+5{,}000^2]$
5000	
7625000000000	$[=3\times 20{,}000^2\times 5{,}000+3\times 20{,}000\times 5{,}000^2+5{,}000^3]$

Seeing that 7,625,000,000,000 is contained in 7,783,426,589,234, *we set down* 5 as the second root-figure. The known part of

the root is now $(20{,}000+5{,}000=)25{,}000$. Putting y for the unknown part, we have $\sqrt[3]{15{,}783{,}426{,}589{,}234}=25{,}000+y$; $15{,}783{,}426{,}589{,}234=(25{,}000+y)^3=25{,}000^3+3\times 25{,}000^2\times y+3\times 25{,}000\times y^2+y^3$. Subtracting $25{,}000^3$ from $15{,}783{,}426{,}589{,}234$, we obtain $158{,}426{,}589{,}234=3\times 25{,}000^2\times y+3\times 25{,}000\times y^2+y^3$.

[This subtraction is most easily performed when we take $7{,}625{,}000{,}000{,}000$ from $7{,}783{,}426{,}589{,}234$: $25{,}000^3=(20{,}000+5{,}000)^3=20{,}000^3+3\times 20{,}000^2\times 5{,}000+3\times 20{,}000\times 5{,}000^2+5{,}000^3=20{,}000^3+7{,}625{,}000{,}000{,}000$; $15{,}783{,}426{,}589{,}234-20{,}000^3=7{,}783{,}426{,}589{,}234$; $7{,}783{,}426{,}589{,}234-7{,}625{,}000{,}000{,}000=158{,}426{,}589{,}234$.]

As $3\times 25{,}000^2\times y$ is much larger than $3\times 25{,}000\times y^2+y^3$, the division of $158{,}426{,}589{,}234$ by $3\times 25{,}000^2$, or of $158{,}426{,}589{,}234$ by $1{,}875{,}000{,}000$, or of $158{,}426$ by $1{,}875$, will give a quotient not much greater than y. Finding that $1{,}875$ is not contained 100 times in $158{,}426$, we write 0 as the hundreds' root-figure. Next, finding that $1{,}875$ is contained more than 80 times, but less than 90 times, in $158{,}426$, we conclude that the tens' figure of the root cannot be higher than 8, which we try:—

$$
\begin{array}{rl}
75080 & [=3\times 25{,}000+80] \\
80 & \\
\hline
6006400 & [=3\times 25{,}000\times 80+80^2] \\
1875000000 & [=3\times 25{,}000^2] \\
\hline
1881006400 & [=3\times 25{,}000^2+3\times 25{,}000\times 80+80^2] \\
80 & \\
\hline
150480512000 & [=3\times 25{,}000^2\times 80+3\times 25{,}000\times 80^2+80^3]
\end{array}
$$

As $150{,}480{,}512{,}000$ is contained in $158{,}426{,}589{,}234$, we set down 8 as the tens' figure of the root. The known part of the root is now $(25{,}000+80=)$ $25{,}080$. Putting z for the part still unknown, we have $\sqrt[3]{15{,}783{,}426{,}589{,}234}=25{,}080+z$; $15{,}783{,}426{,}589{,}234=(25{,}080+z)^3=25{,}080^3+3\times 25{,}080^2\times z+3\times 25{,}080\times z^2+z^3$; and (subtracting $25{,}080^3$ from $15{,}783{,}426{,}589{,}234$) $7{,}946{,}077{,}234=3\times 25{,}080^2\times z+3\times 25{,}080\times z^2+z^3$.

[The subtraction of $25{,}080^3$ from $15{,}783{,}426{,}589{,}234$ is most easily performed when we take $150{,}480{,}512{,}000$ from $158{,}426{,}589{,}234$: $25{,}080^3=(25{,}000+80)^3=25{,}000^3+3\times 25{,}000^2\times 80+3\times 25{,}000\times 80^2+80^3=25{,}000^3+150{,}480{,}512{,}000$; $15{,}783{,}426{,}589{,}234-25{,}000^3=158{,}426{,}589{,}234$; $158{,}426{,}589{,}234-150{,}480{,}512{,}000=7{,}946{,}077{,}234$.]

As $3 \times 25{,}080^2 \times z$ is very much larger than $3 \times 25 \times 080 \times z^2 + z^3$, the division of 7,946,077,234 by $3 \times 25{,}080^2$—or of 7,946,077,234 by 1,887,019,200, or of 79460772 by 18870192—will give either the units' figure of the root or a figure very little higher. The quotient-figure is 6, which we try:—

75244	$[=3 \times 25{,}080+4]$
4	
300976	$[=3 \times 25{,}080 \times 4+4^2]$
1887019200	$[=3 \times 25{,}080^2]$
1887320176	$[=3 \times 25{,}080^2+3 \times 25{,}080 \times 4+4^2]$
4	
7549280704	$[=3 \times 25{,}080^2 \times 4+3 \times 25{,}080 \times 4^2+4^3]$

Seeing that 7,549,280,704 is contained in 7,946,077,234, we write 4 as the units' figure of the root; and finding that 7,549,280,704 is less than 7,946.077,234 by 396,796,530, we conclude that 25,084 is the cube root (not of 15,783,426,589,234, but) of the difference between 15,783.426,589,234 and $25{,}084^3$:— $25{,}084^3 = (25{,}080+4)^3 = 25{,}080^3 + 3 \times 25{,}080^2 \times 4 + 3 \times 25{,}080 \times 4^2 + 4^3 = 25{,}080^3 + 7{,}549{,}280{,}704$; $15{,}783{,}426{,}589{,}234 - 25{,}080^3 = 7{,}946{,}077{,}234$; $7{,}946{,}077{,}234 - 7{,}549{,}280{,}704 = 396{,}796{,}530$. So that $\sqrt[3]{15{,}783{,}426{,}589{,}234}$ lies between 25,084 and 25,085.

Here is the work, both in full and in a contracted form, of the preceding examples:—

Example I.

Uncontracted Process.

157	185193	50
7	125000	+7
1099	60193	
7500	60193	
8599		
7		
60193		

Contracted Process.

157	185193	57
7	125	
1099	60193	
75	60193	
8599		
7		
60193		

EXAMPLE II.

Uncontracted Process.

1220		77308776	400
20		64000000	+20
			+6
24400		13308776	
480000		10088000	
504400		3220776	
20	1266	3220776	
	6		
10088000			
	7596		
	529200		
	536796		
	6		
	3220776		

Contracted Process.

122		77308776	426
2		64	
244		13308	
48		10088	
5044		3220776	
2	1266	3220776	
	6		
10088			
	7596		
	5292		
	536796		
	6		
	3220776		

Example III.

Uncontracted Process.

```
        65000                              15783426589234 | 20,000
         5000                               8000000000000 | +5,000
    ---------                              -------------- |    +80
    325000000                               7783426589234 |     +4
   1200000000                               7625000000000
   ----------                               -------------
   1525000000              75080             158426589234
         5000                 80             150480512000
-------------            -------             ------------
7625000000000            6006400               7946077234
                      1875000000               7549280704
                      ----------               ----------
                      1881006400                396796530
                              80
        75244        ------------
            4        150480512000
       ------
       300976
   1887019200
   ----------
   1887320176
            4
   ----------
   7549280704
```

Contracted Process.

```
  65                                15783426589234 | 25084
   5                                 8
 ---                                ---
 325        7508                     7783
 12            8                     7625
 ----       ----                     ----
 1525      60064                      158426589
    5     187500                      150480512
 ----     --------                   ---------
 7625     18810064        75244       7946077234
                 8            4       7549280704
          --------       ------       ----------
         150480512       300976        396796530
                       18870192
                     ----------
                     1887320176
                              4
                     ----------
                     7549280704
```

It thus appears that a cube root (like a square root) is extracted figure by figure; that, after the finding of the first or *most left-hand* figure, a cube root (like a square root) is, during its *extraction*, regarded as consisting of two parts—the "known" *and the "unknown"*; that, as the work progresses, the known

part of the root becomes gradually larger, and the unknown part gradually smaller; that, every time a new root-figure is obtained, the given number is diminished by the cube of the part of the root then known; that the resulting remainder consists principally of the product of 3 times the square of the known by the unknown part of the root; and that, consequently, the division of the remainder by 3 times the square of the known part of the root must give the rest of the root, or a number not much larger—according as the remainder does or does not contain (*a*) 3 times the product of the known part of the root by the square of the quotient, and (*b*) the cube of the quotient, in addition to 3 times the product of the square of the known part by the quotient.

In the contracted process, we merely dispense with unnecessary ciphers, and (instead of writing the remainders in full) bring down the figures according as they are required—that is, in "periods" of three each: no use being made of the last three figures until the tens' figure of the root has been determined; no use being made of the last six figures until the hundreds' figure of the root has been determined; and so on, as we proceed to the left. (See § 226.) When, therefore, we place a dot over every third figure, beginning with the units' figure, we are able at once to see what particular period is to be brought down at each stage of the work: we are also able to tell, beforehand, how many places the root will occupy—a place for every dot.

238. To extract the Cube Root of a (whole) number: Divide the figures into periods of three each, by placing a dot over every third figure, beginning with the units' figure; and should either one figure or two remain on the extreme left, regard this portion of the number as a period also. Find the highest figure whose cube is contained in the most left-hand period, and set it down as the most left-hand figure of the required root. Subtract the cube of this figure from the first period, and to the remainder annex the second period: the number thus formed will be the first "dividend." Convert this dividend into a "trial"-dividend, by conceiving the last two figures cut off; and, for "trial-divisor," take 3 times the square of the root-figure. Divide the trial-dividend by the *trial-divisor*, and set down the resulting *quotient-figure, upon trial*, as the second figure of the

root. Then, to 3 times the first root-figure annex the quotient-figure, and multiply the resulting number by the quotient-figure; to the product add the trial-divisor—removed two places to the left; and multiply the sum by the quotient-figure. If the number so obtained be contained in the dividend, retain the quotient-figure as the second figure of the root; if not, try a lower quotient-figure. Having found, and written as the second root-figure, the highest figure which, employed in the manner just described, gives a result not greater than the dividend, subtract the result from the dividend, and to the remainder annex the third period: the number so formed will be the second dividend. Convert this new dividend into a new trial-dividend, by conceiving the last two figures cut off; and, for a new trial-divisor, take 3 times the square of the part of the root now known. Treat this pair of numbers—the new trial-dividend and trial-divisor—in the same manner as the preceding pair; and continue the process so long as any period remains to be brought down. The result so obtained will be the cube root of the given number—when the last subtraction leaves no remainder; and will, when there is a remainder, be the cube root of the difference between the given number and this remainder.

Thus, in the case of the last example (contracted process), we begin by dividing the figures of the given number into five periods—the two figures (15) on the extreme left being regarded as a period. For the first root-figure, we set down 2, whose cube (8) is the largest contained in 15. Subtracting 2^3, or 8, from 15, and bringing down the next period (783), we have, as the first "dividend," 7783. Conceiving the last two figures (83) of this dividend cut off, we obtain the first "trial"-dividend, 77. For "trial-divisor," we treble the square of the first root-figure: $2^2 \times 3 = 4 \times 3 = 12$. Finding that 12 is contained 6 times in 77, we try 6, as follows: annexing 6 to the treble of the first root-figure, and multiplying the result (66) by 6, we obtain 396, to which we add the trial-divisor (12)—removed two places to the left; multiplying the sum (1596)

by 6, we find that the product, 9576, is not contained in 7783; for which reason the second root-figure must be lower than 6. Substituting 5 for 6, and finding that 7625 is contained in 7783, we write 5 as the second root-figure. Subtracting 7625 from 7783, and bringing down the next period (426), we have, as the second dividend, 158426. Conceiving the last two figures (26) of this dividend cut off, we obtain the second trial-dividend, 1584. For the second trial-divisor, we treble the square of the known part of the root: $25^2 \times 3 = 625 \times 3 = 1875$. As 1875 is not contained in 1584, we write 0 as the third root-figure. Bringing down the next period (589), we have, as the third dividend, 158426589. Conceiving the last two figures (89) of this dividend cut off, we obtain the third trial-dividend, 1584265. For the third trial-divisor, we treble the square of the known part of the root: $250^2 \times 3 = 62500 \times 3 = 187500$.* Finding that 187500 is contained 8 times, but not 9 times, in 1584265, we try 8 for the next root-figure: annexing 8 to the treble of the known part of the root, and multiplying the result (7508) by 8, we obtain 60064, to which we add the trial-divisor (187500)—removed two places to the left; multiplying the sum (18810064) by 8, we have for product 150480512, which is contained in 158426589. We therefore write 8 as the next root-figure. Subtracting 150480512 from 158426589, and bringing down the last period (234), we have, as the fourth dividend, 7946077234. Conceiving the last two figures (34) of this dividend cut off, we obtain the fourth trial-dividend, 79460772. For the fourth trial-divisor, we treble the square of the known part of the root: $2508^2 \times 3 = 6290064 \times 3 = 18870192$. As 18870192 is contained 4 times, but not 5 times, in 79460772, the units' figure of the root cannot be higher than 4, which we try: annexing 4 to the treble of the known part of the root, and multiplying the result (75244) by 4, we obtain 300976, to which we add the trial-divisor (18870192)—removed two places to the left; multiplying the sum (1887320176) by 4, we have for product 7549280704, which is contained in 7946077234. We therefore set down 4 as the units' figure of the root; and as the subtraction of 7549280704 from 7946077234 gives 396796530 for remainder, 25084 is the cube root (not of 15783426589234, but) of the difference between 15783426589234 and 396796530.

NOTE I.—When a trial-divisor is not contained in the corresponding trial-dividend, write a cipher as the next figure of the root; convert the trial-divisor into a new trial-divisor, by

* *This trial-divisor is most easily obtained when we annex two ciphers to the second trial-divisor* (1875).

annexing two ciphers; convert the dividend into a new dividend, by bringing down the next period; and convert this new dividend into a new trial-dividend by conceiving the last two figures cut off.

NOTE 2.—When the "discovered" figures of a cube root are greater in number, by 2, than those remaining to be discovered, the latter can be found by division only, as follows: For dividend, bring down the outstanding periods, and cut off twice as many figures as there are root-figures to be determined; for divisor, treble the square of the known part of the root; the resulting quotient—the remainder being disregarded—will be the rest of the root.

Thus, in extracting $\sqrt[3]{1290641673199856O897069}$, we need not work, in the ordinary way, for more than the first *five* root-figures (23456)—knowing that the total number is *eight* $\left(\frac{23+1}{3}\right)$. After the finding of the fifth root-figure (6), we have merely to bring down the outstanding periods—all but the last *six* figures; to divide 1302329182560, the number so obtained, by $(23456^2 \times 3 =)\,1650551808$; and, disregarding the remainder, to write 789—the resulting quotient—as the rest of the root:

			1290641673199856O897069	23456
			8	
63			4906	
3			4167	
189	694		739416	
12	4		645904	
1389	2776	7025	93512731	
3	1587	5	82309625	
4167	161476	35125	11203106998	
	4	164268	9900777816	
	645904	16461925	1302329182	
		5		
70356				
6		82309625		
422136				
16497075				
1650129636				
6				
9900777816				

$23456^2 \times 3 = 550183936 \times 3 = 1650551808$;

$1302329182560 \div 1650551808 = 789$; required root $= 23456789$.

The reason is this: Putting x for the last three figures of the root, we have $\sqrt[3]{1290641673199856089 7069}=23456000+x$; $12906416731998560897069=(23456000+x)^3=23456000^3+3\times23456000^2\times x+3\times23456000\times x^2+x^3$. Subtracting 23456000^3, therefore, from 12906416731998560897069, we obtain $1302329182560897069=3\times23456000^2\times x+3\times23456000\times x^2+x^3$; $\frac{1302329182560897069}{3\times23456000^2}=x+\frac{3\times23456000\times x^2}{3\times23456000^2}+\frac{x^3}{3\times23456000^2}=x+\frac{x^2}{23456000}+\frac{x^3}{3\times23456000^2}$. As the numbers represented by x occupies only *three* places, that represented by x^2 occupies not more than *six* places (§ 230), and is therefore less than 1,000,000; so that $\frac{x^2}{23456000}$ is less than $\frac{1,000,000}{23,456,000}$. But $\frac{1,000,000}{23,456,000}$ is less than $\frac{1,000,000}{10,000,000}$: because 23,456,000 occupies eight places, and the smallest (whole) number which occupies eight places is 10,000,000. Consequently, $\frac{x^2}{23456000}$ is less than $\frac{1,000,000}{10,000,000}$, or than $\frac{1}{10}$. Again: the number represented by x^3 occupies not more than *nine* places (§ 237), and is therefore less than 1,000,000,000; so that $\frac{x^3}{3\times23456000^2}$ is less than $\frac{1,000,000,000}{3\times23456000^2}$. But $\frac{1,000,000,000}{3\times23456000^2}$ is less than $\frac{1,000,000,000}{100,000,000,000,000}$: because the square of 23456000 occupies at least fifteen places (§ 230), and the smallest (whole) number which occupies fifteen places is 100,000,000,000,000. Consequently, $\frac{x^3}{3\times23456000^2}$ is less than $\frac{1,000,000,000}{100,000,000,000,000}$, or than $\frac{1}{100,000}$. It thus appears that $\frac{x^2}{23456000}+\frac{x^3}{3\times23456000^2}$ is less than $\frac{1}{10}+\frac{1}{100,000}$, and therefore than unity. So that the division of 1302329182560897069 by 3×23456000^2—or of 1302329182560897069 by 1650551808000000, or of 1302329182560 by 1650551808—gives x, the remainder being disregarded.

239. As the cube of a fraction is obtained when the cube of the numerator is divided by the cube of the denominator—so, the cube root of a fraction is obtained when the cube root of the numerator is divided by the cube root of the denominator.

Thus, $\left(\frac{2}{3}\right)^3=\frac{2}{3}\times\frac{2}{3}\times\frac{2}{3}=\frac{2\times2\times2}{3\times3\times3}=\frac{2^3}{3^3}=\frac{8}{27}$; $\left(\frac{3}{4}\right)^3=\frac{3}{4}\times\frac{3}{4}\times\frac{3}{4}=\frac{3\times3\times3}{4\times4\times4}=\frac{3^3}{4^3}=\frac{27}{64}$; &c.: $\sqrt[3]{\frac{8}{27}}=\frac{\sqrt[3]{8}}{\sqrt[3]{27}}=\frac{2}{3}$; $\sqrt[3]{\frac{27}{64}}=\frac{\sqrt[3]{27}}{\sqrt[3]{64}}=\frac{3}{4}$; &c.

240. The cube root of a fraction whose terms are not cube numbers, or the cube roots of whose terms cannot be found by inspection, is obtained with comparative facility when we multiply the numerator by the square of the denominator, and divide the denominator into the cube root of the product.

Thus, $\sqrt[3]{\frac{4}{5}}=\sqrt[3]{\frac{4\times5^2}{5\times5^2}}=\sqrt[3]{\frac{4\times5^2}{5^3}}=\frac{\sqrt[3]{4\times5^2}}{5}$; $\sqrt[3]{\frac{5}{7}}=\sqrt[3]{\frac{5\times7^2}{7\times7^2}}=\sqrt[3]{\frac{5\times7^2}{7^3}}=\frac{\sqrt[3]{5\times7^2}}{7}$; &c. As a general rule, however, the cube root of a fraction is most easily extracted after the fraction has been converted into a decimal (§ 241).

241. To extract the Cube Root of a number in which a Decimal occurs: Treat the number as an integer—having (if necessary) first annexed a cipher or two, in order to make the number of decimal places a multiple of 3; and remove the decimal point in the resulting root a place to the left for every three decimal places in the given number.

As an illustration, let it be required to extract $\sqrt[3]{\cdot8}$. Although $\sqrt[3]{8}$ is 2, $\sqrt[3]{\cdot8}$ is not ·2; the cube of ·2 being (·2 × ·2 × ·2 =) ·008. We can write ·8 under any of the following forms: $\frac{8}{10}$, $\frac{80}{100}$, $\frac{800}{1,000}$, $\frac{8,000}{10,000}$, $\frac{80,000}{100,000}$, $\frac{800,000}{1,000,000}$, $\frac{8,000,000}{10,000,000}$, $\frac{80,000,000}{100,000,000}$, $\frac{800,000,000}{1,000,000,000}$, &c. Rejecting the fractions $\left(\frac{8}{10}, \frac{80}{100}, \frac{8,000}{10,000}, \frac{80,000}{100,000}, \frac{8,000,000}{10,000,000}, \frac{80,000,000}{100,000,000}, \text{\&c.}\right)$ whose denominators are not cube numbers, we have $\sqrt[3]{\cdot8}=\sqrt[3]{\frac{800}{1,000}}$, or $\sqrt[3]{\frac{800,000}{1,000,000}}$, or $\sqrt[3]{\frac{800,000,000}{1,000,000,000}}$, &c.; $\sqrt[3]{\cdot8}=\frac{\sqrt[3]{800}}{10}$, or

$\frac{\sqrt[3]{800,000}}{100}$, or $\frac{\sqrt[3]{800,000,000}}{1,000}$, &c.; $\sqrt[3]{\cdot 8}=\frac{9}{10}$, or $\frac{92}{100}$, or $\frac{929}{1,000}$, &c.; $\sqrt[3]{\cdot 8}=\cdot 9$, or $\cdot 92$, or $\cdot 929$, &c. In practice, ·8 would be written ·800, or ·800,000, or ·800,000,000, &c.—according to the degree of accuracy considered necessary: in the extraction of its cube root, the number would then be treated as an integer, and the decimal point placed before the first root-figure (9).

As a second illustration, let it be required to extract $\sqrt[3]{6543\frac{8}{17}}$. Converting $\frac{8}{17}$ into a decimal, we have $\sqrt[3]{6543\frac{8}{17}}=$ $\sqrt[3]{6543\cdot 470}=\sqrt[3]{\frac{6543470}{1,000}}=\frac{\sqrt[3]{6543470}}{10}=\frac{187}{10}=18\cdot 7$. A closer approximation is obtained when the decimal representing $\frac{8}{17}$ is carried to six places: $\sqrt[3]{6543\frac{8}{17}}=\sqrt[3]{6543\cdot 470588}=$ $\sqrt[3]{\frac{6543470588}{1,000,000}}=\frac{\sqrt[3]{6543470588}}{100}=\frac{1871}{100}=18\cdot 71$. A still closer approximation is obtained when the decimal representing $\frac{8}{17}$ is carried to nine places: $\sqrt[3]{6543\frac{8}{17}}=\sqrt[3]{6543\cdot 470588235}=$ $\sqrt[3]{\frac{6543470588235}{1,000,000,000}}=\frac{\sqrt[3]{6543470588235}}{1,000}=\frac{18712}{1,000}=18\cdot 712$. In practice, the given number would be written 6543·470, or 6543·470588, or 6543·470588235, &c.—the decimal being made to occupy three places for every figure required in the decimal part of the root: the number would then be treated as an integer, and the decimal point placed before 7—the first root-figure obtained after the bringing down of the first period (470) from the decimal part of the given number.

242. To find an Approximation which shall differ from the Cube Root of a number by less than a given Fractional Unit: Multiply the number by the cube of the denominator of the fractional unit, and divide the denominator into the cube root of the product.

Thus, to find an approximation which shall differ from $\sqrt[3]{57}$ by less than $\frac{1}{12}$, we multiply 57 by 12^3, and divide 12 into the cube root of the product: $\sqrt[3]{57}=\sqrt[3]{\frac{57\times 12^3}{12^3}}=\sqrt[3]{\frac{57\times 1728}{12^3}}$ $=\sqrt[3]{\frac{98496}{12^3}}=\frac{\sqrt[3]{98496}}{12}$. As $\sqrt[3]{98496}$ lies between 46 and 47,

$\frac{\sqrt[3]{98496}}{12}$ lies between $\frac{46}{12}$ and $\frac{47}{12}$; and as the last two fractions differ by exactly $\frac{1}{12}$, each differs from $\sqrt[3]{57}$ by less than $\frac{1}{12}$.

Again, to find an approximation which shall differ from $\sqrt[3]{4567}$ by less than $\frac{1}{1,000}$, we multiply 4567 by $1,000^3$, and divide 1,000 into the cube root of the product: $\sqrt[3]{4567} = \sqrt[3]{\frac{4567 \times 1,000^3}{1,000^3}} = \sqrt[3]{\frac{4567,000,000,000}{1000^3}} = \frac{\sqrt[3]{4567,000,000,000}}{1,000}$.
As $\sqrt[3]{4567,000,000,000}$ lies between 16593 and 16594, $\frac{\sqrt[3]{4567,000,000,000}}{1,000}$ must lie between $\frac{16593}{1,000}$ and $\frac{16594}{1,000}$; and as the last two fractions differ by exactly $\frac{1}{1,000}$, $\sqrt[3]{4567}$ does not differ from either 16·593 or 16·594 by so much as $\frac{1}{1,000}$, or ·001.

Instead of saying that an approximation is to differ from a required root by less than $\frac{1}{1,000}$, we usually say that the root is to be "true" to three decimal places; and, in practice, 4567 would be written under the form 4567·000,000,000—the resulting approximation differing from the truth by less than what the digit 1 would represent in the last decimal place of the approximation.

243. Knowing how to extract square roots and cube roots, we can (in a round-about way) extract such other roots as the 4th, 6th, 8th, 9th, 12th, &c.—in fact, any root indicated by a number which contains no prime factor different from 2 and 3. For instance:

The square root of the square root	of a number is the		4th	root of the number	
The cube root of the square root	"	"	6th	"	"
The square root of the 4th root	"	"	8th	"	"
The cube root of the cube root	"	"	9th	"	"
The cube root of *the 4th root*	"	"	12th	"	"
&c.			&c.		

Thus, the square root of x^4 is x^2, whose square root, x, is the *4th root of x^4*; the square root of x^6 is x^3, whose cube root, x,

is the 6th root of x^6; the 4th root of x^8 is x^2, whose square root, x, is the 8th root of x^8; the cube root of x^9 is x^3, whose cube root, x, is the 9th root of x^9; the 4th root of x^{12} is x^3, whose cube root, x, is the 12th root of x^{12}; &c.

244. *Fractional Indices.*—The employment of fractional indices enables us to write roots under the form of powers—a very convenient form in many cases. Thus, instead of—

$$\left.\begin{array}{l}\sqrt{a}\\ \sqrt[3]{b}\\ \sqrt[4]{c}\\ \sqrt{x^3}\\ \sqrt[3]{y^2}\end{array}\right\}\text{ we can write }\left\{\begin{array}{l}a^{\frac{1}{2}}\\ b^{\frac{1}{3}}\\ c^{\frac{1}{4}}\\ x^{\frac{3}{2}}\\ y^{\frac{2}{3}}\end{array}\right.$$

&c. &c.

Because (§ 220) the square of $a^{\frac{1}{2}}$ is a; the cube of $b^{\frac{1}{3}}$ is b; the fourth power of $c^{\frac{1}{4}}$ is c; the square of $x^{\frac{3}{2}}$ is x^3; the cube of $y^{\frac{2}{3}}$ is y^2; &c.: $(a^{\frac{1}{2}})^2=a^{\frac{1}{2}}\times a^{\frac{1}{2}}=a^{\frac{1}{2}+\frac{1}{2}}=a^1=a$; $(b^{\frac{1}{3}})^3=b^{\frac{1}{3}}\times b^{\frac{1}{3}}\times b^{\frac{1}{3}}=b^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}}=b^1=b$; $(c^{\frac{1}{4}})^4=c^{\frac{1}{4}}\times c^{\frac{1}{4}}\times c^{\frac{1}{4}}\times c^{\frac{1}{4}}=c^{\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}}=c^1=c$; $(x^{\frac{3}{2}})^2=x^{\frac{3}{2}}\times x^{\frac{3}{2}}=x^{\frac{3}{2}+\frac{3}{2}}=x^3$; $(y^{\frac{2}{3}})^3=y^{\frac{2}{3}}\times y^{\frac{2}{3}}\times y^{\frac{2}{3}}=y^{\frac{2}{3}+\frac{2}{3}+\frac{2}{3}}=y^2$; &c.

The expressions $a^{\frac{1}{2}}$, $b^{\frac{1}{3}}$, and $c^{\frac{1}{4}}$ would be read "a in the power $\frac{1}{2}$," "b in the power $\frac{1}{3}$," and "c in the power $\frac{1}{4}$," respectively. Such an expression as $x^{\frac{3}{2}}$ or $y^{\frac{2}{3}}$—the numerator of the index not being unity—could be read in either of two ways: $x^{\frac{3}{2}}$ is "the square root of the cube of x," or "the cube of the square root of x;" and $y^{\frac{2}{3}}$ is "the cube root of the square of y," or "the square of the cube root of y." It has already been shown that $x^{\frac{3}{2}}$ is the square root of x^3, and $y^{\frac{2}{3}}$ the cube root of y^2; and it can be shown, quite as easily, that $x^{\frac{3}{2}}$ is the cube of $x^{\frac{1}{2}}$, and $y^{\frac{2}{3}}$ the square of $y^{\frac{1}{3}}$: $(x^{\frac{1}{2}})^3=x^{\frac{1}{2}}\times x^{\frac{1}{2}}\times x^{\frac{1}{2}}=x^{\frac{1}{2}+\frac{1}{2}+\frac{1}{2}}=x^{\frac{3}{2}}$; $(y^{\frac{1}{3}})^2=y^{\frac{1}{3}}\times y^{\frac{1}{3}}=y^{\frac{1}{3}+\frac{1}{3}}=y^{\frac{2}{3}}$.

PROGRESSION.

245. By a PROGRESSION is meant—a series of three or more numbers which successively increase, or successively decrease, at some uniform rate.

246. The numbers forming a progression are *called its* TERMS, of which the first and the last are

known as the *extremes*, and the intermediate terms as the *means*.

247. A progression is said to be ARITHMETICAL (or equidifferent) when, of every three consecutive terms, the difference between the first and the second is equal to the difference between the second and the third; GEOMETRICAL (or equirational), when, of every three consecutive terms, the ratio which the first bears to the second is equal to the ratio which the second bears to the third; and HARMONICAL, when, of every three consecutive terms, the first bears to the third the same ratio which the difference between the first and the second bears to the difference between the second and the third.

248. A progression is called an *ascending* or a *descending* one—according as the terms increase or decrease from left to right.

ARITHMETICAL PROGRESSION.

249. Being given the first term and the common difference, we can form an arithmetical progression by continually adding the common difference to, or continually subtracting it from, the first term—according as the progression is "ascending" or "descending."

EXAMPLE I.—Set down the ascending arithmetical progression which has 1 for its first term, and 2 for common difference.

Adding 2 to 1, we obtain the second term, 3; adding 2 to 3, we obtain the third term, 5; adding 2 to 5, we obtain the fourth term, 7; and so on. The progression, therefore, is—

1 3 5 7 9 11 13 15 17 19 21 23 &c.

EXAMPLE II.—Set down the descending arithmetical progression which has 35 for its first term, and 3 for common difference.

Subtracting 3 from 35, we obtain the second term, 32; subtracting 3 from 32, we obtain the third term, 29; subtracting 3 from 29, we obtain the fourth term, 26; and so on. The progression, therefore, is—

35 32 29 26 23 20 17 14 11 8 5 2

From an examination of the preceding examples it will be

seen that, in the case of an ascending arithmetical progression, the second term exceeds the first by the common difference; that the third term exceeds the first by twice the common difference; that the fourth term exceeds the first by 3 times the common difference; and so on: and that, in the case of a descending arithmetical progression, the first term exceeds the second by the common difference; that the first term exceeds the third by twice the common difference; that the first term exceeds the fourth by 3 times the common difference; and so on.*

250. To find any term (after the first) of an arithmetical progression: Subtract 1 from the number indicating the place of the term; multiply the remainder by the common difference; and add the product to, or take it from, the first term—according as the progression is an ascending or a descending one.

EXAMPLE III.—The first term of an ascending arithmetical progression is 10, and the common difference 7; what is the 13th term?

The 13th term exceeds the first term by 12 times the common difference. Adding 12×7, therefore, to 10, we find the required term to be 94:—

$$13-1=12; \; 12 \times 7=84; \; 10+84=94.$$

EXAMPLE IV.—The first term of a descending arithmetical progression is 158, and the common difference 9; what is the 11th term?

The 11th term is less than the first term by 10 times the common difference. Subtracting 10×9, therefore, from 158, we find the required term to be 68:—

$$11-1=10; \; 10 \times 9=90; \; 158-90=68.$$

From what we have just seen, it is evident that the difference between the extremes of an arithmetical progression—whether the progression be an ascending or a descending one—is the product of two factors, namely: (*a*) the common difference, and (*b*) what remains when the number of terms is diminished by 1.

251. To find the number of terms in an arith-

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* Putting f for the first term, and d for the common difference, we have, in the case of—

an ascending progression,	*a descending progression,*
1st term $= f$	1st term $= f$
2nd „ $= f+d$	2nd „ $= f-d$
3rd „ $= f+2d$	3rd „ $= f-2d$
4th „ $= f+3d$	4th „ $= f-3d$
nth „ $= f+(n-1)d$	nth „ $= f-(n-1)d$

metical progression—the extremes and the common difference being given: Divide the common difference into the difference between the extremes, and add 1 to the quotient.

EXAMPLE V.—The extremes of an arithmetical progression are 7 and 43, and the common difference is 4; find the number of terms.

The difference between the extremes is 36, in which the common difference (4) is contained 9 times. The number of terms, therefore, is $(9+1=)10$—the terms being more numerous, by 1, than the common differences:—

$$43-7=36;\ 36\div 4=9;\ 9+1=10.$$

252. To find the common difference in an arithmetical progression—the extremes and the number of terms being given: Subtract 1 from the number of terms, and divide the remainder into the difference between the extremes.

EXAMPLE VI.—The extremes of an arithmetical progression are 26 and 2, and the number of terms is 9; find the common difference.

The number of terms being 9, the number of common differences is $(9-1=)8$; and the sum of the common differences is $(26-2=)24$. Dividing 24 by 8, therefore, we find the required common difference to be 3:—

$$26-2=24;\ 9-1=8;\ 24\div 8=3.$$

253. To determine the means in an arithmetical progression — the extremes and the *number* of means being given: Find the common difference (§252), remembering that the number of terms is greater by 2 than the number of means; and then proceed in the manner already explained (§ 249).

EXAMPLE VII.—The extremes of an arithmetical progression are 11 and 39, and the number of means is 6; find the means.

The number of means being 6, the number of terms is $(6+2=)8$. The common difference, therefore (§ 252), is $\frac{39-11}{8-1}=\frac{28}{7}=4$; so that (§249) the required means are $(11+4=)15$, $(15+4=)19$, $(19+4=)23$, $(23+4=)27$, $(27+4=)31$, and $(31+4=)35$. The progression in full is—

11 15 19 23 27 31 35 39

EXAMPLE VIII.—Insert 4 arithmetical means between 37 and 7.

The number of means being 4, the number of terms is $(4+2=)\,6$. The common difference, therefore (§ 252), is $\frac{37-7}{6-1}=\frac{30}{5}=6$; so that (§ 249) the required means are $(37-6=)\,31$, $(31-6=)\,25$, $(25-6=)\,19$, and $(19-6=)\,13$. The progression in full is—

37 31 25 19 13 7

EXAMPLE IX.—Insert one arithmetical mean between 13 and 27.

There being only one mean, the number of terms is $(1+2=)3$. The common difference, therefore, is $\frac{27-13}{3-1}=\frac{14}{2}=7$; so that the required mean is $(13+7=)20$.

EXAMPLE X.—Insert one arithmetical mean between 23 and 15.

As in the last case, the number of terms is $(1+2=)3$; so that the common difference is $\frac{23-15}{3-1}=\frac{8}{2}=4$. The required mean, therefore, is $(23-4=)19$.

ONE arithmetical mean, however, can be inserted more easily.

254. To insert ONE arithmetical mean between two numbers: Add the numbers together, and take half their sum.

Thus, the arithmetical mean between 13 and 27 (Ex. IX.) is $\frac{13+27}{2}=\frac{40}{2}=20$; and the arithmetical mean between 23 and 15 (Ex. X.) is $\frac{23+15}{2}=\frac{38}{2}=19$.

In order to understand this, let us put x for the arithmetical mean between 13 and 27, and suppose that a boy has 13 marbles in his left-hand pocket, and 27 in his right-hand pocket. Now, as 13, x, and 27 form an ascending arithmetical progression (x being greater than 13 by as much as 27 is greater than x), it is evident that, by transferring from his right-hand to his left-hand pocket a number of marbles equal to the common difference, the boy would have x marbles in each pocket. We thus find $x+x=13+27$; $2x=13+27$; $x=\frac{13+27}{2}$.

Next, let us put y for the arithmetical mean between 23 and 15, and suppose that a boy has 23 marbles in his left-hand

pocket, and 15 in his right-hand pocket. As 23, y, and 15 form a descending arithmetical progression (23 exceeding y by as much as y exceeds 15), it is obvious that, by transferring from his left-hand to his right-hand pocket a number of marbles equal to the common difference, the boy would have y marbles in each pocket. Consequently, $y+y=23+15$; $2y=23+15$; $y=\frac{23+15}{2}$.

255. The sum of the extremes of an arithmetical progression is the same as the sum of any two means equally distant from them—the same, for instance, as the sum of the first mean and the last; the same as the sum of the first mean but one and the last but one; &c.

As an illustration, let us take the ascending progression—

1 3 5 7 9 11 13 15 17 19 21 23

If we suppose that a boy has 1 marble in his left-hand pocket, and 23 in his right-hand pocket, we shall see that the transfer of 2 marbles (2 being the common difference) from the right-hand to the left-hand pocket would leave 21 marbles in the right-hand, and 3 in the left-hand pocket; that the transfer of two more in the same direction would leave 19 in the right-hand, and 5 in the left-hand pocket; that another such transfer would leave 17 in the right-hand, and 7 in the left-hand pocket; that another such transfer would leave the contents of the pockets 15 and 9, respectively; and another, 13 and 11, respectively. Hence, $1+23=3+21=5+19=7+17=9+15=11+13$.

As a second illustration, let us take the descending arithmetical progression—

53 48 43 38 33 28 23 18 13 8 3

Supposing that a boy has 53 marbles in his left-hand, and 3 in his right-hand pocket, we see that, by transferring 5 (the common difference) a number of times from the left-hand to the right-hand pocket, he would have — first, 48 in the left-hand, and 8 in the right-hand pocket; next, 43 in the left-hand, and 13 in the right-hand pocket; then, 38 in the left-hand, and 18 in the right-hand pocket; then, 33 in the left-hand, and 23 in the right-hand pocket; and, lastly, 28 in each pocket. Hence, $53+3=48+8=43+13=38+18=33+23=28+28$.

From this last illustration it will be seen that the middle

term of an arithmetical progression which contains an odd number of terms is half the sum of the extremes; and that the middle one of any three consecutive terms is half the sum of the other two.

256. To find the sum of the terms of an arithmetical progression: Multiply the sum of the extremes by half the number of terms.

This follows from § 255. The sum of any two terms equally distant from the extremes being the same as that of the extremes themselves, all the terms, taken together, contain as many such sums as there are *pairs* of terms, and contain, in addition, the half of such a sum when the number of terms is odd—the middle term being, as we have seen, half the sum of the extremes.

Thus, the sum of the terms of the progression—

1 3 5 7 9 11 13 15 17 19 21 23

can be found when the extremes and the number of terms are known. For, from what has already been explained, it is evident that, the sum of the extremes being $(1+23=)24$, all the terms, taken together, contain as many *twenty-fours* as there are *pairs* of terms; so that, the number of terms being 12, and the number of pairs 6, the sum of all the terms is $24 \times 6 = 144$.

Let us next take the progression—

53 48 43 38 33 28 23 18 13 8 3

The sum of the extremes being $(53+3=)56$, and the number of terms 11, the sum of all the terms is $56 \times \frac{11}{2} = 56 \times 5\frac{1}{2} = 308$. Because the middle term (28) is half the sum of the extremes; and the remaining terms, taken together,—being 10 in number,—contain 5 such sums.

257. To find the number of terms in an arithmetical progression—the sum of the extremes and the sum of all the terms being given: Divide the sum of all the terms by the sum of the extremes, and double the quotient.

This follows from § 256. The sum of all the terms being the product of the sum of the extremes by half the number of terms, we obtain half the number of terms on dividing the sum of all the terms by the sum of the extremes. Thus, if the sum of all the terms were 3502, and the sum of the extremes 412, half the number of terms would be $(3502 \div 412 =)\ 8\frac{1}{2}$, and the number of terms $(8\frac{1}{2} \times 2 =)\ 17$.

258. To find either extreme of an arithmetical *progression*—the other extreme, the number of terms,

and the sum of the terms being given: Divide the sum of the terms by half the number of terms, and subtract the given extreme from the quotient.

This, also, follows from § 256. The division of the sum of the terms by half the number of terms gives the sum of the extremes for quotient, which, therefore, exceeds the given extreme by the required extreme. Thus, if the sum of the terms were 1,886, the number of terms 23, and one of the extremes 5, the sum of the extremes would be $(1,886 \div \frac{23}{2} =)$ 164, and the required extreme $(164-5=)$ 159.

259. The sum of the terms of the progression

1 3 5 7 9 &c.

is the square of the number denoting how many terms there are.

Thus, the sum of the first two terms $(1+3)$ is the square of 2; the sum of the first three terms $(1+3+5)$, the square of 3; the sum of the first four terms $(1+3+5+7)$, the square of 4; and so on.*

GEOMETRICAL PROGRESSION.

260. Every term (after the first) of a geometrical progression is the product of the preceding term by the common ratio—the progression being an "ascending" or a "descending" one according as the common ratio is greater or less than unity.

EXAMPLE I.—Set down the geometrical progression which has 5 for its first term, and 2 for common ratio.

Multiplying 5 by 2, we obtain the second term, 10; multiplying 10 by 2, we obtain the third term, 20; multiplying 20 by 2, we obtain the fourth term, 40; and so on. The progression, therefore, is—

5 10 20 40 80 160 320 640 1280 2560 &c.

EXAMPLE II.—Set down the geometrical progression which has 81 for its first term, and $\frac{1}{3}$ for common ratio.

* Putting n for the number of terms, and remembering that the common difference is 2, we have $1+(n-1)\times 2=$ the last term (§ 250); so that (§ 256) the sum of n terms is $\left\{1+1+(n-1)\times 2\right\}\times\frac{n}{2}=\left\{2+2n-2\right\}\times\frac{n}{2}=2n\times\frac{n}{2}=n^2$.

Multiplying 81 by $\frac{1}{3}$, we obtain the second term, 27; multiplying 27 by $\frac{1}{3}$, we obtain the third term, 9; multiplying 9 by $\frac{1}{3}$, we obtain the fourth term, 3; and so on. The progression, therefore, is—

$$81 \quad 27 \quad 9 \quad 3 \quad 1 \quad \tfrac{1}{3} \quad \tfrac{1}{9} \quad \tfrac{1}{27} \quad \tfrac{1}{81} \quad \text{\&c.}$$

The preceding progressions—of which the first is an ascending, and the second a descending one—might be written as follows:

$$5 \quad 5\times2 \quad 5\times2^2 \quad 5\times2^3 \quad 5\times2^4 \quad 5\times2^5 \quad 5\times2^6 \quad 5\times2^7 \quad \text{\&c.}$$

$$81 \quad 81\times\tfrac{1}{3} \quad 81\times(\tfrac{1}{3})^2 \quad 81\times(\tfrac{1}{3})^3 \quad 81\times(\tfrac{1}{3})^4 \quad 81\times(\tfrac{1}{3})^5 \quad \text{\&c.}$$

So that every term (after the first) of a geometrical progression is the product of the first term by a power of the common ratio, the index of which power is less by 1 than the number denoting the place of the term.*

261. The first term and the common ratio being given, to find any other term of a geometrical progression: Subtract one from the number denoting the place of the required term; raise the common ratio to the power indicated by the remainder; and multiply the result by the first term.

EXAMPLE III.—The first term of a geometrical progression is 11, and the common ratio 5; find the seventh term.

Seventh term=first term × sixth power of common ratio= $11\times5^6=11\times15{,}625=171{,}875$.

EXAMPLE IV.—The first term of a geometrical progression is 256, and the common ratio $\frac{3}{4}$; find the tenth term.

Tenth term=first term × ninth power of common ratio= $256\times(\frac{3}{4})^9=256\times\frac{19683}{262144}=19\frac{227}{1024}$.

262. The extremes and the common ratio being given, to find the number of terms in a geometrical progression: Divide the first extreme by the last; find what power of the common ratio the quotient is; and add 1 to the index of this power.

EXAMPLE V.—The extremes of a geometrical progression are 234,375 and 3, and the common ratio is 5; find the number of terms.

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* Putting f for the first term, and r for the common ratio, we have—

2nd term $=f\times r$
3rd „ $=f\times r^2$
4th „ $=f\times r^3$
nth „ $=f\times r^{n-1}$

The first term is the product of the last by a power of the common ratio, the index of which power is less by 1 than the number of terms. We therefore divide 234,375 by 3, and find —in either of the two ways shown below—what power of 5 the quotient (78,125) is. The index of the power being 7, the number of terms is (7+1=)8 :—*

5)78125	5
5)15625	5
5)3125	25
5)625	5
5)125	125
5)25	5
5)5	625
1	5
	3125
	5
	15625
	5
	78125

263. The extremes of a geometrical progression and the number of terms being given, to find the common ratio: Divide the last extreme by the first, and evolve from the quotient the root indicated by what remains when the number of terms is diminished by 1.

---

* We obtain this result almost—if not quite—as readily by actually calculating the intermediate terms, in the way shown in the margin; so that § 262 is *practically* useful only when the number of terms is large, and a table of Logarithms can be employed.

3
5
15
5
75
5
375
5
1875
5
9375
5
46875
5
234375

EXAMPLE VI.—The extremes of a geometrical progression are 6 and 486, and the number of terms is 5; find the common ratio.

The number of terms being 5, the last extreme is the product of the first extreme by the fourth power of the common ratio. So that the common ratio is the fourth root of the fraction $\frac{6}{486}$, or of $\frac{1}{81}$. We obtain this root by extracting the square root of the square root of $\frac{1}{81}$: $\sqrt{\frac{1}{81}}=\frac{1}{9}$; $\sqrt{\frac{1}{9}}=\frac{1}{3}$, the common ratio required.

264. To determine the means in a geometrical progression—the extremes and the *number* of means being given: Find the common ratio (§ 263), remembering that the number of terms is greater by 2 than the number of means; and employ this common ratio, as multiplier, the necessary number of times (§ 260).

EXAMPLE VII.—Insert 5 geometrical means between 3 and 12,288.

The number of means being 5, the number of terms is $(5+2=)7$; so that 12,288 is the product of 3 by the sixth power of the common ratio. To find the common ratio, therefore, we divide 12,288 by 3, and extract the sixth root—or the cube root of the square root—of the quotient: $12{,}288\div3=4{,}096$; $\sqrt{4{,}096}=64$; $\sqrt[3]{64}=4$, the common ratio. Multiplying 3 (the first extreme) by 4, we obtain the second term, 12; multiplying 12 by 4, we obtain the third term, 48; and so on. The required means are thus found to be 12, 48, 192, 768, and 3,072. The progression in full is—

3 12 48 192 768 3,072 12,288.

EXAMPLE VIII.—Insert one geometrical mean between 539 and 11.

The number of terms being $(1+2=)3$, the last extreme is the product of the first extreme by the second power of the common ratio; so that the common ratio is the square root of the fraction $\frac{11}{539}$, or of $\frac{1}{49}$. The square root of $\frac{1}{49}$ being $\frac{1}{7}$, the required mean is $539\times\frac{1}{7}=77$.

ONE geometrical mean, however, can be inserted more easily.

265. To insert ONE geometrical mean between two numbers: Multiply the numbers together, and extract the square root of the product.

Thus, putting x for the geometrical mean between 539 and 11, we have the proportion—

$$539 : x :: x : 11.$$

Consequently, $x^2=539\times11$; $x=\sqrt{539\times11}=\sqrt{5929}=77$.

NOTE.—From this it follows that, of any three consecutive terms of a geometrical progression, the middle term is the square root of the product of the other two.

266. To find the sum of the terms of a geometrical progression—the extremes and the common ratio being given: Multiply the last extreme by the common ratio, and divide the difference between the product and the first extreme by the difference between the common ratio and unity.

Thus, putting s for the sum of the terms of the progression 5 20 80 320 1,280 5,120 20,480; multiplying by the common ratio, 4; taking equals from equals; and dividing by 3 (the difference between the common ratio and 1), we have—

$$s = 5 + 20 + 80 + 320 + 1{,}280 + 5{,}120 + 20{,}480$$
$$4s = \quad 20 + 80 + 320 + 1{,}280 + 5{,}120 + 20{,}480 + 81{,}920$$

$$3s = 81{,}920 - 5\,;\ s = \frac{81{,}920 - 5}{3} = 27{,}305.$$

Again: putting s for the sum of the terms of the progression 9 3 1 $\frac{1}{3}$ $\frac{1}{9}$ $\frac{1}{27}$ $\frac{1}{81}$ $\frac{1}{243}$ $\frac{1}{729}$; multiplying by the common ratio, $\frac{1}{3}$; taking equals from equals; and dividing by $\frac{2}{3}$ (the difference between the common ratio and 1), we have—

$$s = 9 + 3 + 1 + \tfrac{1}{3} + \tfrac{1}{9} + \tfrac{1}{27} + \tfrac{1}{81} + \tfrac{1}{243} + \tfrac{1}{729}$$
$$\tfrac{1}{3} \text{ of } s = \quad 3 + 1 + \tfrac{1}{3} + \tfrac{1}{9} + \tfrac{1}{27} + \tfrac{1}{81} + \tfrac{1}{243} + \tfrac{1}{729} + \tfrac{1}{2{,}187}$$

$$\tfrac{2}{3} \text{ of } s = 9 - \frac{1}{2{,}187}\,;\ s = \frac{9 - \frac{1}{2187}}{\frac{2}{3}} = 13\tfrac{364}{729}.$$

NOTE 1.—When a geometrical progression is descending and interminate, the sum of the terms is obtained from the division of the first term by the difference between the common ratio and unity—the last term being regarded as 0. Thus, writing the pure circulator $\cdot\dot{3}4\dot{5}$ under the form

$$\frac{345}{1{,}000} + \frac{345}{1{,}000{,}000} + \frac{345}{1{,}000{,}000{,}000} + \text{\&c.},$$

and putting s for the sum of all the fractions, we have—

$$s = \frac{345}{1{,}000} + \frac{345}{1{,}000{,}000} + \frac{345}{1{,}000{,}000{,}000} + \text{\&c.}$$
$$\frac{1}{1{,}000} \times s = \qquad \frac{345}{1{,}000{,}000} + \frac{345}{1{,}000{,}000{,}000} + \text{\&c.}$$

$$\frac{999}{1{,}000} \times s = \frac{345}{1{,}000}\,;\ s = \frac{345}{1{,}000} \div \frac{999}{1{,}000} = \frac{345}{999}.$$

NOTE 2.—Every term (after the first) of the progression

1 2 4 8 16 32 &c.

exceeds the sum of the preceding terms by 1. Thus, the second term (2) exceeds the first (1) by 1; the third term (4) exceeds the sum of the first two ($1+2=3$) by 1; the fourth term (8) exceeds the sum of the first three ($1+2+4=7$) by 1; &c.*

The fact is well known to professional gamblers that, after playing for and losing—first (say) £1, then £2, then £4, then £8, and so on, a person would, if he ultimately won a game, retrieve all his losses, and have £1 over.

HARMONICAL PROGRESSION.

267. Four numbers are said to be in harmonical *proportion* when the first bears to the fourth the same ratio which the difference between the first and the second bears to the difference between the third and the fourth.

Thus, 12, 6, 15, and 10 are in harmonical proportion, 12 bearing to 10 the same ratio which the difference between 12 and 6 bears to the difference between 15 and 10:

$$12 : 10 :: 12-6 : 15-10.$$

268. Three numbers are said to be in harmonical *proportion* when the first bears to the third the same ratio which the difference between the first and the second bears to the difference between the second and the third.

Thus, 12, 8, and 6 are in harmonical proportion, 12 bearing to 6 the same ratio which the difference between 12 and 8 bears to the difference between 8 and 6:

$$12 : 6 :: 12-8 : 8-6.$$

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* The common ratio being 2, the *n*th term is $(1 \times 2^{n-1}=)2^{n-1}$, and the sum of *n* terms is $\left(\frac{2^{n-1} \times 2 - 1}{2-1} = \right) 2^n - 1$; whilst the next term after the *n*th is $(1 \times 2^n =) 2^n$, which exceeds $2^n - 1$ by 1.

269. A harmonical PROGRESSION—as already explained (§ 247)—is a series of numbers of which every three consecutive ones are in harmonical proportion.

Thus, 120, 60, 40, 30, 24, and 20 form a harmonical progression, every three consecutive numbers being in harmonical proportion:

$$\begin{aligned} 120 : 40 &:: 120-60 : 60-40 \\ 60 : 30 &:: 60-40 : 40-30 \\ 40 : 24 &:: 40-30 : 30-24 \\ 30 : 20 &:: 30-24 : 24-20 \end{aligned}$$

270. To insert a harmonical mean between two numbers: Divide the sum of the numbers into twice their product.

Thus, taking any three consecutive terms of the harmonical progression

120 60 40 30 24 20,

we find that the middle one is obtained as quotient when twice the product of the other two is divided by their sum:

$$60 = \frac{2 \times 120 \times 40}{120+40};\ 40 = \frac{2 \times 60 \times 30}{60+30};\ \&c.$$

Putting y for the harmonical mean between x and z, we have $x : z :: x-y : y-z$; $xy-xz=xz-yz$; $xy+yz=xz+xz$; $(x+z) \times y=2xz$; $y=\frac{2xz}{x+z}$.

271. By taking the reciprocals* of its terms, we convert a harmonical into an arithmetical progression, or an arithmetical into a harmonical progression—as the case may be.

Thus, taking the reciprocals of the terms of the harmonical progression

120 60 40 30 24 20,

we obtain the arithmetical progression

$$\tfrac{1}{120} \quad \tfrac{1}{60} \quad \tfrac{1}{40} \quad \tfrac{1}{30} \quad \tfrac{1}{24} \quad \tfrac{1}{20}.$$

---

* Any two numbers whose product is *unity* are said to be the "reciprocals" of one another. Thus, $\frac{2}{3}$ and $\frac{3}{2}$ are reciprocals of one another, $\frac{2}{3} \times \frac{3}{2}$ being $= 1$; 7 and $\frac{1}{7}$ are also reciprocals of one another, $7 \times \frac{1}{7}$ being $= 1$. So that, to find the reciprocal of a number, we divide *unity by the* number. The reciprocal of $5\frac{1}{2}$, for instance, or of $\frac{11}{2}$, is $(1 \div \frac{11}{2} =) \frac{2}{11}$.

On the other hand, taking the reciprocals of the terms of the arithmetical progression

$$3 \quad 5 \quad 7 \quad 9 \quad 11,$$

we obtain the harmonical progression

$$\tfrac{1}{3} \quad \tfrac{1}{5} \quad \tfrac{1}{7} \quad \tfrac{1}{9} \quad \tfrac{1}{11}.$$

Let x, y, and z be three consecutive terms of a harmonical progression, and let their reciprocals be x', y', and z', respectively: to prove that x', y', and z' are three consecutive terms of an arithmetical progression; in other words (§ 254), that $x'+z'=2y'$. Here we have (§ 270) $xy+yz=2xz$; (dividing by xyz) $\frac{xy}{xyz}+\frac{yz}{xyz}=\frac{2xz}{xyz}$; $\frac{1}{z}+\frac{1}{x}=\frac{2}{y}=2\times\frac{1}{y}$; *i.e.*, $z'+x'=2y'$.

Next, let x', y', and z' be three consecutive terms of an arithmetical progression, and let their reciprocals be x, y, and z, respectively: to prove that x, y, and z are three consecutive terms of a harmonical progression; in other words (§ 270), that $y=\frac{2xz}{x+z}$. Here we have (§ 254) $x'+z'=2y'$; (dividing by $x'y'z'$) $\frac{x'}{x'y'z'}+\frac{z'}{x'y'z'}=\frac{2y'}{x'y'z'}$; $\frac{1}{y'z'}+\frac{1}{x'y'}=\frac{2}{x'z'}$; $\frac{1}{y'}\times\frac{1}{z'}+\frac{1}{x'}\times\frac{1}{y'}=2\times\frac{1}{x'}\times\frac{1}{z'}$; *i.e.*, $y\times z+x\times y=2\times x\times z$; $(z+x)\times y=2xz$; $y=\frac{2xz}{z+x}$.

272. To insert two or more harmonical means between a given pair of extremes: Between the reciprocals of the extremes insert as many arithmetical means as there are harmonical means to be determined; and the reciprocals of the arithmetical means will be the harmonical means required.

This follows from § 271. As an illustration, let it be required to insert three harmonical means between 36 and 4. The reciprocals of the given numbers are $\frac{1}{36}$ and $\frac{1}{4}$. Between these two fractions we insert (§ 253) three arithmetical means—

$$\tfrac{1}{12} \quad \tfrac{5}{36} \quad \tfrac{7}{36},$$

the reciprocals of which are the harmonical means required—

$$12 \quad \tfrac{36}{5} \quad \tfrac{36}{7}$$

$$\text{or } 12 \quad 7\tfrac{1}{5} \quad 5\tfrac{1}{7}$$

As a second illustration, let it be required to set down four

additional terms—two on the left, and two on the right—of the harmonical progression

$$40 \qquad 30 \qquad 24 \qquad 20$$

Taking the reciprocals of the given numbers, we obtain the arithmetical progression

$$\tfrac{1}{40} \qquad \tfrac{1}{30} \qquad \tfrac{1}{24} \qquad \tfrac{1}{20}$$

The common difference being $\frac{1}{120}$, we extend this progression as follows: (left-hand side) $\frac{1}{40} - \frac{1}{120} = \frac{1}{60}$; $\frac{1}{60} - \frac{1}{120} = \frac{1}{120}$; (right-hand side) $\frac{1}{20} + \frac{1}{120} = \frac{7}{120}$; $\frac{7}{120} + \frac{1}{120} = \frac{8}{120} = \frac{1}{15}$. The arithmetical progression, in its extended form, being thus found to be

$$\tfrac{1}{120}, \quad \tfrac{1}{60} \quad \tfrac{1}{40}, \quad \tfrac{1}{30} \quad \tfrac{1}{24} \quad \tfrac{1}{20} \quad \tfrac{7}{120} \quad \tfrac{1}{15},$$

we obtain the harmonical progression, in its extended form, by taking the reciprocals of these fractions:

$$120 \qquad 60 \qquad 40 \qquad 30 \qquad 24 \qquad 20 \qquad 17\tfrac{1}{7} \qquad 15.$$

NOTE.—A harmonical progression is said to be so called from the circumstance that musical strings of equal thickness and tension must, in order to produce *harmony* when sounded together, vary in length as the numbers

$$1 \qquad \tfrac{1}{2} \qquad \tfrac{1}{3} \qquad \tfrac{1}{4} \qquad \tfrac{1}{5} \qquad \&c.$$

These numbers, it will be seen, form a harmonical progression, and are the reciprocals of the series of "natural" numbers

$$1 \qquad 2 \qquad 3 \qquad 4 \qquad 5 \qquad \&c.$$

273. If, between any two numbers, regarded as extremes, there were inserted (I.) a harmonical mean, (II.) a geometrical mean, and (III.) an arithmetical mean, those means—taken in the order in which they have been mentioned—would form three consecutive terms of a geometrical progression.

Between 4 and 9, for instance, the harmonical mean is $\left(\frac{2\times4\times9}{4+9}=\right) 5\frac{7}{13}$; the geometrical mean, $(\sqrt{4\times9}=\sqrt{36}=)$ 6; and the arithmetical mean, $\left(\frac{4+9}{2}=\right) 6\frac{1}{2}$;—and $5\frac{7}{13}$, 6, and $6\frac{1}{2}$ are in continued proportion:

$$5\tfrac{7}{13} : 6 :: 6 : 6\tfrac{1}{2}$$

If the extremes were x and y, the harmonical mean would be $\frac{2xy}{x+y}$; the geometrical mean, $\sqrt{xy}$; and the arithmetical mean, $\frac{x+y}{2}$;—and it is obvious that $\frac{2xy}{x+y}$, $\sqrt{xy}$, and $\frac{x+y}{2}$ would form three consecutive terms of a geometrical progression:

$$\frac{2xy}{x+y} : \sqrt{xy} :: \sqrt{xy} : \frac{x+y}{2}$$

LOGARITHMS.

274. By means of what is termed a "Table of LOGARITHMS" we are able to substitute addition for multiplication, subtraction for division, multiplication for involution, and division for evolution.

We can set about the construction of such a table by simply writing the arithmetical progression

$$0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad \&c.$$

above any geometrical progression whose first term is unity. Thus, selecting 7 for common ratio, we have—

0	1	2	3	4	5	6	7	8	&c.
1	7	49	343	2,401	16,807	117,649	823,543	5,764,801	&c.

These two sets of numbers afford us a ready means of, for instance, (*a*) multiplying 49 by 16,807; (*b*) dividing 823,543 by 2,401; (*c*) raising 2,401 to the second power; and (*d*) extracting the cube root of 117,649:

(*a*) Above 49 and 16,807 stand 2 and 5, respectively; the sum of 2 and 5 is 7; and below 7 stands 823,543—the product of 49 and 16,807.

(*b*) Above 823,543 and 2,401 stand 7 and 4, respectively; the difference between 7 and 4 is 3; and below 3 stands 343—the quotient resulting from the division of 823,543 by 2,401.

(*c*) Above 2,401 stands 4; the double of 4 is 8; and below 8 stands 5,764,801—the square of 2,401.

(*d*) Above 117,649 stands 6; one-third of 6 is 2; and below 2 stands 49—the cube root of 117,649.

In order to understand the reason of this, we have merely to reflect that the terms of the geometrical progression are all *powers* of 7, and that the terms of the arithmetical progression are the *indices* of those powers, respectively: $1=7^0$ (§ 222); $7=7^1$; $49=7^2$; $343=7^3$; $2,401=7^4$; &c.:—

$$(a)\ 49\times 16,807=7^2\times 7^5=7^7=823,543$$
$$(b)\ 823,543\div 2,401=7^7\div 7^4=7^3=343$$
$$(c)\ 2,401^2=(7^4)^2=7^8=5,764,801$$
$$(d)\ \sqrt[3]{117,649}=\sqrt[3]{7^6}=7^2=49.$$

It will be observed that the number of *sevens* which occur, as factors, in any particular term of the geometrical progression is indicated by the corresponding term of the arithmetical progression. This explains the word LOGARITHM, which literally means "number of [common] ratios." In the case under con-

sideration, the logarithm of 1 is 0; of 7, 1; of 49, 2; of 343, 3; &c.—the common ratio, 7, being recognised as the BASE of the table.

We now proceed to construct a different table, by taking (say) 4 for common ratio, instead of 7:

0	1	2	3	4	5	6	7	8	9	10	&c.
1	4	16	64	256	1,024	4,096	16,384	65,536	262,144	1,048,576	&c.

A glance at this table enables us to say that, for instance, (*a*) the product of 256 and 4,096 is 1,048,576; (*b*) the division of 262,144 by 16,384 gives 16 for quotient; (*c*) the third power of 16 is 4,096; and (*d*) the square root of 1,048,576 is 1,024:

(*a*) Above 256 and 4,096 stand 4 and 6, respectively; the sum of 4 and 6 is 10; and below 10 stands 1,048,576—the product of 256 and 4,096.

(*b*) Above 262,144 and 16,384 stand 9 and 7, respectively; the difference between 9 and 7 is 2; and below 2 stands 16—the quotient obtained when 262,144 is divided by 16,384.

(*c*) Above 16 stands 2; the treble of 2 is 6; and below 6 stands 4,096—the cube of 16.

(*d*) Above 1,048,576 stands 10; the half of 10 is 5; and below 5 stands 1,024—the square root of 1,048,576.

This is explained by the fact that the terms of the geometrical progression are all *powers* of 4, and that the terms of the arithmetical progression are the *indices* of those powers, respectively: $1=4^0$; $4=4^1$; $16=4^2$; $64=4^3$; $256=4^4$; &c.:—

(*a*) $256\times 4{,}096=4^4\times 4^6=4^{10}=1{,}048{,}576$
(*b*) $262{,}144\div 16{,}384=4^9\div 4^7=4^2=16$
(*c*) $16^3=(4^2)^3=4^6=4{,}096$
(*d*) $\sqrt{1{,}048{,}576}=\sqrt{4^{10}}=4^5=1{,}024$.

In this last table, each term of the arithmetical progression indicates the "number of [common] ratios"—that is, the number of *fours*—which occur, as factors, in the corresponding term of the geometrical progression. So that the logarithm of 1 is 0; of 4, 1; of 16, 2; of 64, 3; &c.—the common ratio, 4, being recognised as the "base" of the table.

It must be borne in mind that two different tables—or "systems"—of logarithms cannot be employed indiscriminately. As an illustration of this, let us endeavour, by means of the two tables now before us, to find (*a*) the product of 343 and 256; also, (*b*) the number of times 16 is contained in 5,764,801:

0	1	2	3	4	5	6	7	8	&c.
1	*7*	*49*	*343*	2,401	16,807	117,649	823,543	5,764,801	&c.
1	*4*	*16*	*64*	256	1,024	4,096	16,384	65,536	&c.

(*a*) The logarithms of 343 and 256 are 3 and 4, respectively —but *not to the same base*; the sum of 3 and 4 is 7, the logarithm of 823,543 (to the base 7), and also of 16,384 (to the base 4); neither 823,543 nor 16,384, however, is the product of 343 and 256:

$343 \times 256 = 7^3 \times 4^4$; and $7^3 \times 4^4$ is neither 7^7 nor 4^7—*i.e.*, is neither 823,543 nor 16,384.

(*b*) The logarithms of 16 and 5,764,801 are 2 and 8, respectively—but *not to the same base;* the difference between 2 and 8 is 6, the logarithm of 117,649 (to the base 7), and also of 4,096 (to the base 4); neither 117,649 nor 4,096, however, expresses the number of times 16 is contained in 5,764,801:

$5{,}764{,}801 \div 16 = 7^8 \div 4^2$; and $7^8 \div 4^2$ is neither 7^6 nor 4^6—*i.e.*, is neither 117,649 nor 4,096.

In constructing a table of logarithms, therefore, we must confine ourselves to some *one* base: moreover, we must devise some means of determining the logarithms of all the numbers intermediate between every two powers of the base. Neither of the preceding tables, for instance, is of any practical utility —the first exhibiting the logarithm of none of the numbers between 1 and 7, between 7 and 49, between 49 and 343, &c.; whilst the second table exhibits the logarithm of none of the numbers between 1 and 4, between 4 and 16, between 16 and 64, &c.

We now proceed to consider how this defect can be remedied. We have already seen (p. 274) that, of any three consecutive terms of a geometrical progression, the middle term is the *square root of the product* of the other two; and (p. 269) that, of any three consecutive terms of an arithmetical progression, the middle term is *half the sum* of the other two. These facts suggest a means by which either of the foregoing tables could be completed. Let us take the table whose base is 4:

0	1	2	3	4	&c.
1	4	16	64	256	&c.

Now, if 16 and 2, for instance, were removed from this table, we could replace both numbers by inserting a geometrical mean between 4 and 64, and an arithmetical mean between 1 and 3: $\sqrt{4 \times 64} = \sqrt{256} = 16$; $\frac{1+3}{2} = \frac{4}{2} = 2$. Again, if 64 and 3 were removed, we could replace both by inserting a geometrical mean between 16 and 256, and an arithmetical mean between 2 and 4: $\sqrt{16 \times 256} = \sqrt{4096} = 64$; $\frac{2+4}{2} = \frac{6}{2} = 3$.

In like manner, by inserting a geometrical mean between 1 and 4, *and an* arithmetical mean between 0 and 1, we find *that the logarithm* of 2 is $\frac{1}{2}$; by inserting a geometrical mean

between 4 and 16, and an arithmetical mean between 1 and 2, we find that the logarithm of 8 is $\frac{3}{2}$; &c.: $\sqrt{1\times 4}=\sqrt{4}=2$; $\frac{0+1}{2}=\frac{1}{2}$; $\sqrt{4\times 16}=\sqrt{64}=8$; $\frac{1+2}{2}=\frac{3}{2}$; &c. :*

0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	3	4	&c.
1	2	4	8	16	64	256	&c.

So that the continual insertion of geometrical means in the lower, and of arithmetical means in the upper series, would ultimately complete the table, by exhibiting the logarithms, not merely of 1, 4, 16, 64, &c., but of all the intermediate numbers also. The arithmetical mean between 0 and $\frac{1}{2}$, for instance, would be the logarithm of the geometrical mean between 1 and 2; the arithmetical mean between $\frac{1}{2}$ and 1 would be the logarithm of the geometrical mean between 2 and 4; and so on.†

The base of the table of logarithms now in universal use is neither 7 nor 4, but 10, which, as we shall presently find, is by far the most appropriate number that could have been selected. The following portion of the table involves no calculation:

0	1	2	3	4	5	6	&c.
1	10	100	1,000	10,000	100,000	1,000,000	&c.

Here the logarithm of 1 is 0; of 10, 1; of 100, 2; of 1,000, 3; &c. Because 0, 1, 2, 3, &c. are the indices of the powers which 1, 10, 100, 1,000, &c., respectively, are of the base 10: $1=10^0$; $10=10^1$; $100=10^2$; $1,000=10^3$; &c. The table could be completed upon the principle just explained. For instance: a being the geometrical mean between 1 and 10, and a' the arithmetical mean between 0 and 1, a' would be the logarithm of a; b being the geometrical mean between 1 and a, and b' the arithmetical mean between 0 and a', b' would be the logarithm of b; c being the geometrical mean between 1 and b,

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* It is evident that $\frac{1}{2}$ and $\frac{3}{2}$ are the logarithms of 2 and 8, respectively—to the base 4. Because when 2 is expressed as a power of 4, the index is $\frac{1}{2}$; and when 8 is so expressed, the index is $\frac{3}{2}$: $2=4^{\frac{1}{2}}$; $8=4^{\frac{3}{2}}$.

† Here is a general demonstration of the fact that *the arithmetical mean between the logarithms of any two numbers is the logarithm of the geometrical mean between the numbers themselves.* Let x and z be two numbers, whose logarithms are a and c, respectively; let y be the geometrical mean between x and z, and b the arithmetical mean between a and c: to prove that b is the logarithm of y.

Putting B for the base of the table, we have (the logarithm of x being a) $x=B^a$, and (the logarithm of z being c) $z=B^c$; $x\times z=B^a\times B^c=B^{a+c}$; *i.e.*, ($x\times z$ being$=y^2$, and $a+c=2b$,) $y^2=B^{2b}$; $y=B^b$; $b=$ *logarithm of* y.

and c' the arithmetical mean between 0 and b', c' would be the logarithm of c; and so on:

0	c'	b'	a'	1	2	3	&c.
1	c	b	a	10	100	1,000	&c.

After the insertion of the necessary number of means, the table would appear under this form:

0	·3010300	·4771213	·6020600	·6989700	·7781513	·8450980	·9030900	·9542425	1	&c.
1	2	3	4	5	6	7	8	9	10	&c.

That is—

$1=10^{0}$	$6=10^{\cdot 7781513}$
$2=10^{\cdot 3010300}$	$7=10^{\cdot 8450980}$
$3=10^{\cdot 4771213}$	$8=10^{\cdot 9030900}$
$4=10^{\cdot 6020600}$	$9=10^{\cdot 9542425}$
$5=10^{\cdot 6989700}$	$10=10^{1}$
&c. &c.	&c. &c.

Or—

Logarithm of		Logarithm of	
1	= 0	6	=·7781513
2	= ·3010300	7	=·8450980
3	= ·4771213	8	=·9030900
4	= ·6020600	9	=·9542425
5	= ·6989700	10	=1
&c.	&c.	&c.	&c.

275. The Logarithm of a number may be defined to be—the index of the power which the number is of 10.

Logarithms were first heard of in the year 1614, when the inventor, Baron Napier of Scotland, published a table having, for base, 2·718281828459—a number not selected arbitrarily, but suggested by the peculiar line of investigation which had led Napier to the construction of his table. Three years afterwards (1617), at the suggestion of a Mr. Briggs, Napier adopted 10 for base. This explains a distinction which is sometimes drawn between the *Napierian* and the *Common* logarithm of a number: the "Napierian" logarithm of a number being the index of the power which the number is of *2·718281828459*; and the "Common" logarithm, the index of *the power which* the number is of 10. When nothing to the

contrary is stated, the word "logarithm" is always understood to mean "COMMON logarithm."*

276. The logarithm of the product of two or more factors is the sum of the logarithms of those factors.

Thus, the logarithms of 2 and 3 being ·3010300 and ·4771213, respectively, the logarithm of 6 (*i.e.*, of the product of 2 and 3)

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* From a theorem known to algebraists as the "Exponential Theorem," Napier deduced the following beautiful formula for the construction of a table of logarithms—to *any* base:

$$\text{Log}\,(n+1)=\log n+2\text{M}\times\left\{\frac{1}{2n+1}+\frac{1}{3(2n+1)^3}+\frac{1}{5(2n+1)^5}+\&\text{c.}\right\}$$

The value of M—which is called the *modulus*—being determined, this formula enables us to convert the logarithm of any number, n, into the logarithm of the next higher number, $n+1$, by the addition of

$$2\,\text{M}\times\left\{\frac{1}{2n+1}+\frac{1}{3(2n+1)^3}+\frac{1}{5(2n+1)^5}+\frac{1}{7(2n+1)^7}+\&\text{c.}\right\}$$

The value of M depends upon the number selected for base. Napier selected 2·718281828459 for base, in order to have M=1; but in the Common system of logarithms, the modulus is ·43429448.

Substituting 1, 2, 3, &c., in succession, for n, and remembering that the logarithm of 1 is 0, we thus have—

NAPIERIAN

$$\text{Log}\,2=\log 1+2\times\left\{\frac{1}{3}+\frac{1}{3^4}+\frac{1}{5\times3^5}+\frac{1}{7\times3^7}+\&\text{c.}\right\}$$
$$=0+{\cdot}6931472 \qquad = {\cdot}6931472$$
$$\text{Log}\,3=\log 2+2\times\left\{\frac{1}{5}+\frac{1}{3\times5^3}+\frac{1}{5^6}+\frac{1}{7\times5^7}+\&\text{c.}\right\} \qquad =1{\cdot}0986123$$
$$\text{Log}\,4=2\times\log 2\ [\S\,278] \qquad =1{\cdot}3862944$$
$$\text{Log}\,5=\log 4+2\times\left\{\frac{1}{9}+\frac{1}{3\times9^3}+\frac{1}{5\times9^5}+\frac{1}{7\times9^7}+\&\text{c.}\right\} \qquad =1{\cdot}6094379$$

&c. &c.

COMMON

$$\text{Log}\,2=\log 1+{\cdot}86858896\times\left\{\frac{1}{3}+\frac{1}{3^4}+\frac{1}{5\times3^5}+\frac{1}{7\times3^7}+\&\text{c.}\right\}$$
$$=0+{\cdot}3010300 \qquad ={\cdot}3010300$$
$$\text{Log}\,3=\log 2+{\cdot}86858896\times\left\{\frac{1}{5}+\frac{1}{3\times5^3}+\frac{1}{5^6}+\frac{1}{7\times5^7}+\&\text{c.}\right\} \qquad ={\cdot}4771213$$
$$\text{Log}\,4=2\times\log 2 \qquad ={\cdot}6020600$$
$$\text{Log}\,5=\log 4+{\cdot}86858896\times\left\{\frac{1}{9}+\frac{1}{3\times9^8}+\frac{1}{5\times9^5}+\frac{1}{7\times9^7}+\&\text{c.}\right\}$$
$$={\cdot}6989700$$

&c &c.

is $\cdot 7781513$—the sum of $\cdot 3010300$ and $\cdot 4771213$: $2 = 10^{\cdot 3010300}$; $3 = 10^{\cdot 4771213}$; $2 \times 3 = 10^{\cdot 3010300} \times 10^{\cdot 4771213} = 10^{\cdot 3010300 + \cdot 4771213} = 10^{\cdot 7781513}$; $\cdot 7781513$=logarithm of $(2 \times 3 =)6$.

The logarithms of x, y, and z being a, b, and c, respectively, $a+b+c$ is the logarithm of the product of x, y, and z: $x = 10^a$, $y = 10^b$, $z = 10^c$; $x \times y \times z = 10^a \times 10^b \times 10^c = 10^{a+b+c}$; *i.e.*, $a+b+c$ = logarithm of $(x \times y \times z)$.

277. The logarithm of a fraction is what remains when the logarithm of the denominator is subtracted from that of the numerator.

Thus, the logarithms of 5 and 4 being $\cdot 69897$ and $\cdot 60206$, respectively, the logarithm of the fraction $\frac{5}{4}$ is $\cdot 69897 - \cdot 60206 = \cdot 09691$: $5 = 10^{\cdot 69897}$; $4 = 10^{\cdot 60206}$; $5 \div 4 = 10^{\cdot 69897} \div 10^{\cdot 60206} = 10^{\cdot 69897 - \cdot 60206} = 10^{\cdot 09691}$; $\cdot 09691$= logarithm of $\frac{5}{4}$.

The logarithms of x and y being a and b, respectively, $a-b$ is the logarithm of $\frac{x}{y}$: $x = 10^a$; $y = 10^b$; $x \div y = 10^a \div 10^b = 10^{a-b}$; *i.e.*, $a-b$=logarithm of $\frac{x}{y}$.

278. When we multiply the logarithm of a number by—

$$\left.\begin{matrix}2\\3\\4\end{matrix}\right\}\text{ we obtain the logarithm of the }\left\{\begin{matrix}\text{2nd power}\\\text{3rd }\ \text{,,}\\\text{4th }\ \text{,,}\end{matrix}\right\}\text{ of the number.}$$
&c. &c.

This follows from § 276. Taking, for instance, the number 7, we have—

$\text{Log } 7^2 = \log (7 \times 7) = \log 7 + \log 7 = 2 \times \log 7$
$\text{Log } 7^3 = \log (7 \times 7 \times 7) = \log 7 + \log 7 + \log 7 = 3 \times \log 7$
$\text{Log } 7^4 = \log (7 \times 7 \times 7 \times 7) = \log 7 + \log 7 + \log 7 + \log 7 = 4 \times \log 7$.
&c. &c.

279. When we divide the logarithm of a number by—

$$\left.\begin{matrix}2\\3\\4\end{matrix}\right\}\text{ we obtain the logarithm of the }\left\{\begin{matrix}\text{square root}\\\text{cube }\ \text{,,}\\\text{fourth }\ \text{,,}\end{matrix}\right\}\text{ of the number.}$$
&c. &c.

This also follows from § 276. Taking, as before, the number 7, we have—

Log $7 = \log(7^{\frac{1}{2}} \times 7^{\frac{1}{2}}) = \log 7^{\frac{1}{2}} + \log 7^{\frac{1}{2}} = 2 \times \log 7^{\frac{1}{2}}$; $\log 7^{\frac{1}{2}} = \frac{1}{2}$ of log 7

Log $7 = \log(7^{\frac{1}{3}} \times 7^{\frac{1}{3}} \times 7^{\frac{1}{3}}) = \log 7^{\frac{1}{3}} + \log 7^{\frac{1}{3}} + \log 7^{\frac{1}{3}} = 3 \times \log 7^{\frac{1}{3}}$; $\log 7^{\frac{1}{3}} = \frac{1}{3}$ of log 7

Log $7 = \log(7^{\frac{1}{4}} \times 7^{\frac{1}{4}} \times 7^{\frac{1}{4}} \times 7^{\frac{1}{4}}) = \log 7^{\frac{1}{4}} + \log 7^{\frac{1}{4}} + \log 7^{\frac{1}{4}} + \log 7^{\frac{1}{4}} = 4 \times \log 7^{\frac{1}{4}}$; $\log 7^{\frac{1}{4}} = \frac{1}{4}$ of log 7

&c. &c.

280. In the great majority of instances, a logarithm is merely an *approximation*—sufficiently close, however, for all practical purposes, and consisting of two parts, an integer and a decimal.

Thus, the powers of 10 being 10, 100, 1,000, &c., it is evident that the intermediate numbers cannot be expressed as powers of 10, except approximately. The logarithm of 16, for instance, is only an approximation; 16 not being a power of 10. As 16 lies between 10 and 100, whose logarithms are 1 and 2, respectively, the logarithm of 16 must lie between 1 and 2, and nearer to 1 than to 2. On consulting a table, we find the logarithm of 16 to be 1·20412; that is, $16 = 10^{1 \cdot 20412} = 10^{\frac{120412}{100000}} = 10^{\frac{30103}{25000}}$. So that if 10 were raised to the 30,103[rd] power, and the 25,000[th] root of this power extracted, the result would be *very nearly* equal to 16.

281. The integral part of a logarithm is termed the CHARACTERISTIC; the decimal part, the MANTISSA.

Thus, 1·20412, the logarithm of 16, has 1 for "characteristic," and ·20412 for "mantissa." A table of logarithms gives the mantissae,* but not the characteristics, which can be found by inspection (§§ 284–5).

282. When two numbers are so related that the larger is the product of the smaller by a power of 10,—or when either can be converted into the other by the removal of the decimal point,—the logarithms of the two have the same mantissa, and differ only in their characteristics.

Thus, the logarithm of 365 being 2·5622929,

Log $3650 = \log(365 \times 10) = \log 365 + \log 10 = 2{\cdot}5622929 + 1 = 3{\cdot}5622929$

Log $36500 = \log(365 \times 100) = \log 365 + \log 100 = 2{\cdot}5622929 + 2 = 4{\cdot}5622929$

* The decimal point, however, is dispensed with—the most left-hand figure of each mantissa being understood to belong to the first decimal place.

$\text{Log}\,365000 = \log(365 \times 1{,}000) = \log 365 + \log 1{,}000 = 2\cdot5622929$
$+3 = 5\cdot5622929$

$\text{Log}\,36\cdot5 = \log\frac{365}{10} = \log 365 - \log 10 = 2\cdot5622929 - 1 = 1\cdot5622929$

$\text{Log}\,3\cdot65 = \log\frac{365}{100} = \log 365 - \log 100 = 2\cdot5622929 - 2 =$
$0\cdot5622929$

$\text{Log}\,\cdot365 = \log\frac{3\cdot65}{10} = \log 3\cdot65 - \log 10 = 0\cdot5622929 - 1 =$
$\bar{1}\cdot5622929$

$\text{Log}\,\cdot0365 = \log\frac{3\cdot65}{100} = \log 3\cdot65 - \log 100 = 0\cdot5622929 - 2 =$
$\bar{2}\cdot5622929$

&c. &c.

NOTE.—The preceding property (§ 282) is peculiar to the Common system of logarithms, and explains the early substitution of this system for the Napierian, in which no two of the logarithms of such numbers as ·0365, ·365, 3·65, 36·5, 365, 3650, 36500, 365000, &c. would be found to have the same mantissa.

283. In determining the mantissa of the logarithm of a number which contains a Decimal, we *treat the number as an integer;* and in determining the mantissa of the logarithm of an integer which ends with one or more ciphers, we *conceive the cipher or ciphers cut off.*

It has just been shown (§ 282) that the logarithms of 36·5, 3·65, ·365, ·0365, 3650, 36500, and 365000 have all the same mantissa—namely, ·5622929—as the logarithm of 365.

284. The characteristic of the logarithm of a number greater than unity is the number indicating how many places the most left-hand digit is from the units' place.*

Thus, the characteristic of the logarithm of—

(*a*)	6	is	0
(*b*)	83·542	,,	1
(*c*)	987	,,	2
(*d*)	7541	,,	3
(*e*)	30456·78	,,	4
	&c.		&c.

Because (*a*) 6 is *no* place from the units' place; (*b*) 8 is *one* place; (*c*) 9, *two* places; (*d*) 7, *three* places; (*e*) 3, *four* places; &c.

* *Such a characteristic, therefore, is less by unity than the number of figures in the integral part of the given number.*

The reason is this: A number greater than unity must, when its most left-hand figure is

$$\left.\begin{array}{l} \textit{no}\text{ place} \\ \textit{one}\ \text{,,} \\ \textit{two}\text{ places} \\ \textit{three}\ \text{,,} \\ \textit{four}\ \text{,,} \\ \text{\&c.} \end{array}\right\} \begin{array}{c}\text{from the units' place,} \\ \text{be less than}\end{array} \left\{\begin{array}{r} 10 \\ 100 \\ 1{,}000 \\ 10{,}000 \\ 100{,}000 \\ \text{\&c.} \end{array}\right\} \begin{array}{c}\text{but not} \\ \text{less than}\end{array} \left\{\begin{array}{r} 1 \\ 10 \\ 100 \\ 1{,}000 \\ 10{,}000 \\ \text{\&c.} \end{array}\right.$$

And it is obvious that when a number is less than

$$\left.\begin{array}{r} 10 \\ 100 \\ 1{,}000 \\ 10{,}000 \\ 100{,}000 \\ \text{\&c.} \end{array}\right\} \text{but not less than} \left\{\begin{array}{r} 1 \\ 10 \\ 100 \\ 1{,}000 \\ 10{,}000 \\ \text{\&c.} \end{array}\right\} \begin{array}{c}\text{the logar-} \\ \text{ithm of} \\ \text{the num-} \\ \text{ber is} \\ \text{less than}\end{array} \left\{\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ \text{\&c.} \end{array}\right\} \text{but not less than} \left\{\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ \text{\&c.} \end{array}\right\} \begin{array}{c}\text{so that} \\ \text{the charac-} \\ \text{teristic is}\end{array} \left\{\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ \text{\&c.} \end{array}\right.$$

The presence of a decimal in a number greater than unity does not at all affect the characteristic of the logarithm of the number. The logarithm of 83·542, for instance, has the same characteristic as the logarithm of 83: because 83·542, although greater than 83, is less than 84, whose logarithm has the same characteristic as the logarithm of 83.

285. The characteristic of the logarithm of a Decimal is the number indicating how many places the most left-hand DIGIT is from the units' place.* Above such a characteristic the sign *minus* (—) is always placed, to denote that the characteristic is *subtractive*—the mantissa being additive.

Thus, the characteristic of the logarithm of—

(*a*) ·567	is	$\overline{1}$
(*b*) ·045	,,	$\overline{2}$
(*c*) ·008	,,	$\overline{3}$
(*d*) ·000706	,,	$\overline{4}$
(*e*) ·0000912	,,	$\overline{5}$
&c.		&c.

Because (*a*) 5 is *one* place from the units' place; (*b*) 4, *two* places; (*c*) 8, *three* places; (*d*) 7, *four* places; (*e*) 9, *five* places; &c.

In order to understand the reason of this, we must remember that every decimal is convertible into a fraction, whose logarithm may be regarded as consisting of two parts—the one additive, the other subtractive; the additive part being the logarithm of the numerator, and the subtractive part the logarithm of the

* So that the characteristic of the logarithm of a decimal is greater *by unity* than the *number of prefixed ciphers* in the decimal.

denominator. For, as "multiplying" by a fraction means multiplying by the numerator and *dividing* by the denominator —so, "adding" the logarithm of a fraction means adding the logarithm of the numerator and *subtracting* the logarithm of the denominator (§§ 276–7.)

A decimal whose most left-hand digit occupies the *first* decimal place is a number like ·7, or ·78, or ·789, &c.; and the logarithm of every such number has $\bar{1}$ for characteristic:

$$\text{Log}\ {\cdot}7 = \log\frac{7}{10} = \log 7 - \log 10 = {\cdot}845098 - 1 = \bar{1}{\cdot}845098$$

$$\text{Log}\ {\cdot}78 = \log\frac{7{\cdot}8}{10} = \log 7{\cdot}8 - \log 10 = {\cdot}892095 - 1 = \bar{1}{\cdot}892095$$

$$\text{Log}\ {\cdot}789 = \log\frac{7{\cdot}89}{10} = \log 7{\cdot}89 - \log 10 = {\cdot}897077 - 1 = \bar{1}{\cdot}897077$$

&c. &c.

A decimal whose most left-hand digit occupies the *second* decimal place is a number like ·07, or ·078, or ·0789, &c.; and the logarithm of every such number has $\bar{2}$ for characteristic:

$$\text{Log}\ {\cdot}07 = \log\frac{7}{100} = \log 7 - \log 100 = {\cdot}845098 - 2 = \bar{2}{\cdot}845098$$

$$\text{Log}\ {\cdot}078 = \log\frac{7{\cdot}8}{100} = \log 7{\cdot}8 - \log 100 = {\cdot}892095 - 2 = \bar{2}{\cdot}892095$$

$$\text{Log}\ {\cdot}0789 = \log\frac{7{\cdot}89}{100} = \log 7{\cdot}89 - \log 100 = {\cdot}897077 - 2 = \bar{2}{\cdot}897077$$

&c. &c.

A decimal whose most left-hand digit occupies the *third* decimal place is a number like ·007, or ·0078, or ·00789, &c.; and the logarithm of every such number has $\bar{3}$ for characteristic:

$$\text{Log}\ {\cdot}007 = \log\frac{7}{1{,}000} = \log 7 - \log 1{,}000 = {\cdot}845098 - 3 = \bar{3}{\cdot}845098$$

$$\text{Log}\ {\cdot}0078 = \log\frac{7{\cdot}8}{1{,}000} = \log 7{\cdot}8 - \log 1{,}000 = {\cdot}892095 - 3 = \bar{3}{\cdot}892095$$

$$\text{Log}\ {\cdot}00789 = \log\frac{7{\cdot}89}{1{,}000} = \log 7{\cdot}89 - \log 1{,}000 = {\cdot}897077 - 3 = \bar{3}{\cdot}897077$$

&c. &c.

Here is a more general explanation: A decimal expressed by n figures is equivalent to a fraction having for denominator 10^n, and for numerator a (whole) number occupying, say, $n-c$ places; c representing the number of *ciphers* which intervene between the decimal point and the most left-hand digit of the decimal. The logarithm of the numerator of this fraction has $n-c-1$ *for characteristic* (§ 284); and the logarithm of the

denominator is n. So that—the mantissa of the logarithm of the numerator being regarded as additive—the logarithm of the fraction, or of the decimal equivalent to the fraction, has $c+1$, marked with the sign *minus*, for characteristic; n exceeding $n-c-1$ by $c+1$. And it will be seen that $c+1$ is the number indicating the position of the most left-hand digit of the decimal, with respect to the units' place.

286. To find, by means of a table, the mantissa of the logarithm of a number: (I.) If the number be expressed by not more than four figures, the mantissa of its logarithm will be found, in the column headed "0," immediately to the right of the number itself, which will be found in the column headed "N." (II.) If the number be expressed by five figures, the mantissa of its logarithm will be found in the column at whose head stands the last figure of the five, and in the same horizontal row as the number—found in the "N" column—expressed by the other four figures. (III.) If the number be expressed by more than five figures, set down the mantissa of the logarithm of the number expressed by the first five (on the left); subtract this mantissa from the next higher one given in the table; multiply the remainder by the additional figure or figures of the given number; and, cutting off as many figures as there are in the multiplier, add the resulting product to the mantissa first obtained.

(I.) The mantissa of the logarithm of

6 is ·7781513
67 ,, ·8260748
678 ,, ·8312297
6,789 ,, ·8318058

These mantissae are found in column "0," immediately to the right of 6, 67, 678, and 6,789, respectively, which occur in column "N."

(II.) The mantissa of the logarithm of

98765 is ·9946031
87654 ,, ·9427717
76543 ,, ·8839055

These mantissae are found in column "5," column "4," and

column "3," respectively, and in the same horizontal row of figures as 9876, 8765, and 7654, respectively, which occur in column "N."

(III.) The mantissa of the logarithm of

(*a*) 234567 is ·3702670
(*b*) 2345678 ,, ·3702684
(*c*) 23456789 ,, ·3702686

These mantissae are obtained as follows:—

$$\text{Mantissa of logarithm of}\begin{cases} 23456 = {\cdot}3702540 \\ 23457 = {\cdot}3702725 \end{cases}$$

$${\cdot}3702725 - {\cdot}3702540 = {\cdot}0000185$$

(*a*) $185[\textit{ten-millionths}] \times 7$ (the last figure of 234567) $= 1295$; $1295 \div 10 = 130$ nearly;

$$\begin{array}{r} {\cdot}3702540 \\ 130 \\ \hline \end{array}$$

$${\cdot}3702540 + {\cdot}0000130 = {\cdot}3702670$$

(*b*) 185×78 (the last two figures of 2345678) $= 14430$; $14430 \div 100 = 144$;

$$\begin{array}{r} {\cdot}3702540 \\ 144 \\ \hline \end{array}$$

$${\cdot}3702540 + {\cdot}0000144 = {\cdot}3702684$$

(*c*) 185×789 (the last three figures of 23456789) $= 145965$; $145965 \div 1{,}000 = 146$ nearly;

$$\begin{array}{r} {\cdot}3702540 \\ 146 \\ \hline \end{array}$$

$${\cdot}3702540 + {\cdot}0000146 = {\cdot}3702686$$

In order to understand the reason of this, we must bear in mind that (§ 282) the logarithms of 234560, 2345600, and 23456000 have the same mantissa as the logarithm of 23456; also, that the logarithms of 234570, 2345700, and 23457000 have the same mantissa as the logarithm of 23457.

(*a*) As 234567 lies between 234560 and 234570, which differ by 10, and whose logarithms differ by 185 ten-millionths, and as 234567 exceeds 234560 by 7, the logarithm of 234567 must exceed the logarithm of 234560 by the fourth term of the proportion—

$$10 : 7 :: 185 : 185 \times 7 \div 10$$

(*b*) As 2345678 lies between 2345600 and 2345700, which differ by 100, and whose logarithms differ by 185 ten-millionths, and as 2345678 exceeds 2345600 by 78, the logarithm of 2345678 must exceed the logarithm of 2345600 by the fourth term of the proportion—

$$100 : 78 :: 185 : 185 \times 78 \div 100$$

U 2

(c) As 23456789 lies between 23456000 and 23457000, which differ by 1,000, and whose logarithms differ by 185 ten-millionths, and as 23456789 exceeds 23456000 by 789, the logarithm of 23456789 must exceed the logarithm of 23456000 by the fourth term of the proportion—

$$1{,}000 : 789 :: 185 : 789 \times 185 \div 1{,}000$$

NOTE.—It is by no means correct to say that the differences between the logarithms of numbers are proportional to the differences between the numbers themselves. Taking, for example, 1,000, 100, and 10, whose logarithms are 3, 2, and 1, respectively, we see that the difference (900) between 1,000 and 100 does *not* bear to the difference (90) between 100 and 10 the same ratio which the difference (1) between 3 and 2 bears to the difference (1) between 2 and 1. Within narrow limits, however, the results obtained in the manner just explained are sufficiently accurate for practical purposes.

287. *Tabular Differences.*—By a "tabular difference" is meant—the difference, as given in a table, between two consecutive logarithms: or rather the *average* of as many such differences as are nearly equal. Column "D.," however, in which tabular differences are found, dispenses with unnecessary ciphers, as well as with the decimal point—the most right-hand figure of a tabular difference being understood to belong to the same place as the most right-hand figures of the mantissae to which the difference appertains.

Thus, ·0000185, the difference between the logarithms of 23456 and 23457, is found in column D., under the form 185.

A glance at a table of logarithms will show that, as we pass from smaller to larger numbers, the tabular differences gradually decrease. This is easily accounted for. The logarithms of 73 and 74, for instance, differ by ·0059088; and if we double 73 and 74, we shall have exactly the same difference between the logarithms of the resulting numbers:

$$\text{Log } 146 = \log (73 \times 2) = \log 73 + \log 2$$
$$\text{Log } 148 = \log (74 \times 2) = \log 74 + \log 2$$
$$\text{Log } 148 - \log 146 = \log 74 - \log 73 = \cdot 0059088.$$

Now, as 147 lies between 146 and 148, the logarithm of 147 *cannot differ* from the logarithm of either 146 or 148 by so *much as* ·0059088.

Let us next multiply 73 and 74 by 5:

$$\text{Log } 365 = \log(73 \times 5) = \log 73 + \log 5$$
$$\text{Log } 370 = \log(74 \times 5) = \log 74 + \log 5$$
$$\text{Log } 370 - \log 365 = \log 74 - \log 73 = \cdot 0059088.$$

Between 365 and 370 there occur the four numbers 366, 367, 368, and 369; so that ·0059088, the difference between the logarithms of 365 and 370, is the sum of five smaller differences, namely—

$$\text{Difference between the logs. of}\begin{cases}(a)\ 365 \text{ and } 366 = \cdot 0011882\\(b)\ 366\ \text{,,}\ \ 367 = \cdot 0011850\\(c)\ 367\ \text{,,}\ \ 368 = \cdot 0011817\\(d)\ 368\ \text{,,}\ \ 369 = \cdot 0011786\\(e)\ 369\ \text{,,}\ \ 370 = \cdot 0011753\end{cases}$$

$$\overline{\cdot 0059088}$$

288. *Proportional Parts.*—The number which, when determining a mantissa not to be found in a table, we add to the next lower mantissa given in a table is called a "proportional part."

As examples of proportional parts, we may take 130, 144, and 146—the increments employed in the conversion of ·3702540 into the mantissae of the logarithms of 234567, 2345678, and 23456789, respectively. (See p. 291.)

289. Under the head "P.P.," or "Pr. Pts.," a table of logarithms gives the proportional part for each of the nine digits: so that the proportional part for any particular combination of figures can be found by addition only—being the sum of the proportional parts for the individual digits of the combination; that is, when those parts are made to occupy the same relative positions as the digits.

For instance, the proportional part, as given in a table, for the last figure of

$$\left.\begin{array}{l}234561\\234562\\234563\\234564\\234565\\234566\\234567\\234568\\234569\end{array}\right\} \text{is} \left\{\begin{array}{r}19\\37\\56\\74\\93\\111\\130\\148\\167\end{array}\right.$$

It will be seen that, as the numbers on the left all lie between 234560 and 234570, which differ by 10, and whose logarithms differ by 185, the numbers on the right are obtained from the following proportions:—

10 : 1 :: 185 : 19 (nearly)
10 : 2 :: 185 : 37
10 : 3 :: 185 : 56 (nearly)
10 : 4 :: 185 : 74
10 : 5 :: 185 : 93 (nearly)
10 : 6 :: 185 : 111
10 : 7 :: 185 : 130 (nearly)
10 : 8 :: 185 : 148
10 : 9 :: 185 : 167 (nearly)

It thus becomes evident that the proportional part for the last figure of

$$\left.\begin{array}{l}2345601\\2345602\\2345603\\2345604\\2345605\\2345606\\2345607\\2345608\\2345609\end{array}\right\}\text{ is the \textit{tenth} part of }\left\{\begin{array}{r}19\\37\\56\\74\\93\\111\\130\\148\\167\end{array}\right.$$

Because, as the left-hand numbers all lie between 2345600 and 2345700, which differ by 100, and whose logarithms differ by 185, the preceding proportions would, if 100 were substituted for 10, give the proportional parts.

It is equally evident that the proportional part for the last figure of

$$\left.\begin{array}{l}23456001\\23456002\\23456003\\23456004\\23456005\\23456006\\23456007\\23456008\\23456009\end{array}\right\}\text{ is the \textit{hundredth} part of }\left\{\begin{array}{r}19\\37\\56\\74\\93\\111\\130\\148\\167\end{array}\right.$$

For, as the left-hand numbers all lie between 23456000 and 23457000, which differ by 1,000, and whose logarithms differ by 185, the preceding proportions would, if 1,000 were substituted for 10, give the proportional parts.

We are now in a position to find, by addition only, the *proportional* part for, say, the last three figures of 23456789. The *proportional* parts for 7, 8, and 9 being 130, $\frac{1}{10}$ of 148, and $\frac{1}{100}$ of

167, respectively, the proportional part for the combination 789 is 130+15+1=146. In practice, 130, 148, and 167 would simply be taken from a table and written in the way shown in the margin—the last two figures (47) of the sum being afterwards rejected.

$$\begin{array}{r} 130 \\ 148 \\ 167 \\ \hline 14647 \end{array}$$

290. To determine, with the aid of a table, a number whose logarithm is given: (I.) Should the mantissa of the logarithm occur in the table, set down the number which, in column N., stands in the same horizontal row as the mantissa; and to this number annex the figure at the head of the column in which the mantissa is found. (II.) Should the mantissa not occur in the table, find the next lower mantissa which does, and set down the number to whose logarithm this lower mantissa belongs; then, subtract the approximate mantissa from the given one; treat the remainder as the numerator of a fraction having the tabular difference for denominator; convert this fraction into a decimal; and, disregarding the decimal point, annex the decimal to the number already set down. The figures of the required number having thus been obtained, the characteristic will determine the position of the decimal point (§§ 284–5).

For instance: (I.) The digits of the number whose logarithm has ·9427717 for mantissa are 87654; the mantissa occurring in column 4, and in the same horizontal row as 8765—found in column N. So that if the characteristic of the logarithm were

0		8·7654
1		87·654
2		876·54
3		8765·4
4		87654
5	the number would be	876540
$\bar{1}$		·87654
$\bar{2}$		·087654
$\bar{3}$		·0087654
$\bar{4}$		·00087654
$\bar{5}$		·000087654
&c.		&c.

Because the characteristic indicates how many places—to the left or right, as the case may be—the first digit (8) must stand from the units' place.

(II.) Next, let it be required to find the digits of the number whose logarithm has, for mantissa, ·3702686, which does not occur in a table. This mantissa lies between ·3702540 and ·3702725—the mantissae, as given in a table, of the logarithms of 23456 and 23457, respectively. We therefore set down 23456 as a portion—and the principal portion—of the required number.

The logarithm of 23457 exceeds the logarithm of 23456 by (·3702725 − ·3702540 =) 185; the logarithm of the required number exceeds the logarithm of 23456 by (·3702686 − ·3702540 =) 146; and 23457 exceeds 23456 by 1. Consequently, the required number exceeds 23456 by the fourth term of the proportion—

$$185 : 146 :: 1 : a$$

$$a = \frac{146}{185} = \cdot 789$$

So that the required number is (23456 + ·789 =) 23456789—the decimal point being disregarded: because the ultimate position of the point must be determined by the characteristic.

When extreme accuracy is not required, the fourth term of such a proportion as the last can be taken from a table, digit by digit; the fact been borne in mind that—in the case of seven-figure mantissae—the proportional part which a table gives for any particular digit is understood to be read

ten-millionths }
hundred-millionths } when the number of other figures preceding the digit is { *five*
billionths } { *six*
&c. { *seven*
&c.

Thus, the given mantissa being greater than that of the logarithm of 23456 by 146, and the highest proportional part which does not exceed 146 being 130—the proportional part for 7, we write 7 as the sixth figure of the required number. This leaves (146 − 130 =) 16 ten-millionths to be compensated for, so to speak. Annexing a cipher to 16, we obtain 160 hundred-millionths; and, the highest proportional part which does not exceed 160 being 148—the proportional part for 8, we write 8 as the seventh figure of the required number. This leaves (160 − 148 =) 12 hundred-millionths to be compensated for. Annexing a cipher to 12, we obtain 120 billionths, for which we more than compensate by writing 7* as the eighth

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* *Although* apparently too high, this figure is really too low—the *proportion* giving 789, instead of 787. Practically, however, the discrepancy would be found immaterial.

and last figure of the required number—the proportional part for 7 being, as we have seen, (not 120, but) 130.

EXAMPLE I.—Find the product of 456700, 23·895, and ·000834756.

Adding together 5·6596310, 1·3783070, and $\bar{4}$·9215595,—the logarithms of 456700, 23·895, and ·000834756, respectively,—we find the logarithm of the required product to be 3·9594975; so that the product itself is 9109·56:

Log 456700=5·6596310
Log 23·895=1·3783070
Log ·000834756=$\bar{4}$·9215595

3·9594975=log 9109·56;
9109·56=456700×23·895×·000834756.

EXAMPLE II.—Divide 31664·1864 by 785·4.

Subtracting 2·8950909, the logarithm of 785·4, from 4·5005684, the logarithm of 31664·1864, we find the logarithm of the required quotient to be 1·6054775; so that the quotient itself is 40·316.

Log 31664·1864=4·5005684
Log 785·4 =2·8950909

1·6054775
1·6054775=log 40·316;
40·316=31664·1864÷785·4.

EXAMPLE III.—Find the fourth term (a) of the proportion

2534·8 : 19327·56 :: 385·694 : a

Adding together 4·2861770 and 2·5862429, the logarithms of the means, we find the logarithm of the product of the means to be 6·8724199, from which we subtract 3·4039437, the logarithm of the given extreme. The remainder, 3·4684762, is the logarithm of the fourth term of the proportion; so that the fourth term (a) itself is 2940·8728.

Log 19327·56 =4·2861770
Log 385·694=2·5862429

6·8724199
Log 2534·8 =3·4039437

3·4684762
3·4684762=log 2940·8728;
2940·8728=a.

NOTE.—In finding the difference between two logarithms, we sometimes begin by adding to both a number which makes the subtrahend 10. The number so added is known as the "*arithmetical complement*" of the subtrahend, and is, of course, the difference between the subtrahend and 10. Thus, the *arithmetical* complement of 3·4039437 being (10—3·4039437=)

6·5960563,* the subtraction of 10 from 13·4684762—that is, of 3·4039437+6·5960563 from 6·8724199+6·5960563—leaves the same remainder as the subtraction of 3·4039437 from 6·8724199. So that the logarithm of the fourth term of the last proportion can be obtained as follows, 10 being subtracted mentally from 13—the integral part of the sum of the three addends:

Log 19327·56 =		4·2861770
Log 385·694=		2·5862429
Log 2534·8	=3·4039437; arith. comp. =	6·5960563
Log (19327·56×385·694÷2534·8)		= 3·4684762

EXAMPLE IV.—Find the 17th power of ·85764.

The logarithm of ·85764 being $\bar{1}$·9333050, the logarithm of ·85764[17] is $\bar{1}$·9333050×17=$\bar{2}$·8661850; so that the required power is ·073483 (nearly).

$$\begin{array}{r} \text{Log } \cdot 85764 = \bar{1}\cdot 9333050 \\ 17 \\ \hline \text{Log } \cdot 85764^{17} = \bar{2}\cdot 8661850 \end{array}$$

$\bar{2}$·8661850=log ·073483;
·073483 =·85764[17].

In multiplying $\bar{1}$·9333050 by 17, we obtain (·9333050×17=) 15·8661850, which is additive, from the mantissa; and ($\bar{1}$×17=) $\bar{17}$, which is subtractive, from the characteristic. We therefore write the product ($\bar{17}$+15·8661850) under the form $\bar{2}$·8661850; $\bar{17}$+15 being equivalent to $\bar{2}$. Because, diminishing a number by 2 is obviously the same as performing the twofold operation of (*a*) adding 15 to the number, and (*b*) subtracting 17 from the result.

EXAMPLE V.—Extract the 7th root of 316·4758.

The logarithm of 316·4758 being 2·5003405, the logarithm of $\sqrt[7]{316\cdot4758}$ is $\frac{1}{7}$ of 2·5003405—*i.e.*, 0·3571915; so that the required root is 2·2761.

Log 316·4758=2·5003405;
2·5003405÷7=0·3571915;
0·3571915=log 2·2761;
2·2761=$\sqrt[7]{316\cdot4758}$

EXAMPLE VI.—Extract the 5th root of ·000000813972.

The logarithm of ·000000813972 being $\bar{7}$·9106095, the logarithm of $\sqrt[5]{\cdot000000813972}$ is $\frac{1}{5}$ of $\bar{7}$·9106095—*i.e.*, $\bar{2}$·7821219; so that the required root is ·060551:

* This is most easily obtained when we subtract the last figure (7) of 3·4039437 from 10, and each of the others from 9—the "carrying" of 1 being dispensed with.

$$\begin{array}{r} \text{Log}\ \cdot000000813972=\bar{7}\cdot9106095 \\ \underline{\bar{3}+3} \\ 5)\overline{\bar{10}+3\cdot9106095} \\ \hline \bar{2}\cdot7821219 \end{array}$$

$$\bar{2}\cdot7821219=\log\cdot060551\,;$$
$$\cdot060551=\sqrt[5]{\cdot000000813972}.$$

The manner in which $\bar{7}\cdot9106095$ is divided by 5 requires a word of explanation. One-fifth of $7\cdot9106095$ being $1\cdot5821219$, one-fifth of $\bar{7}\cdot9106095$ would, at first sight, appear to be $\bar{1}\cdot5821219$; but if we multiply $\bar{1}\cdot5821219$ by 5, we shall obtain ($\bar{5}+2\cdot9106095=$) $\bar{3}\cdot9106095$, instead of $\bar{7}\cdot9106095$. The division of $\bar{7}$ by 5 would give $\bar{1}$ for quotient, and leave $\bar{2}$—equivalent to $\bar{20}$ tenths—for remainder. If 9 tenths were added to this remainder, the result would be (not 29, but) $\bar{11}$ tenths, the division of which by 5 would give $\cdot\bar{2}$ for quotient, and leave $\cdot\bar{1}$ for remainder. So that, if the division were continued, the mantissa would be marked *minus*, as well as the characteristic; whereas A MANTISSA IS NEVER MARKED MINUS* (unless when written, as a subtrahend, after another logarithm).

In every such case as the one under consideration, we begin by so altering the *form* of the logarithm as to have, for characteristic, a *multiple of the divisor;* and we naturally select the lowest multiple which answers our purpose. The first multiple of 5, after 7, being 10, which exceeds 7 by 3, we convert $\bar{7}\cdot9106095$ into $\bar{10}+3\cdot9106095$ by adding $\bar{3}$ to the characteristic, and $+3$ to the mantissa; $\bar{3}+3$ being$=0$.† We then find, with the greatest facility, that $\frac{1}{5}$ of $\bar{7}\cdot9106095$ is $\bar{2}\cdot7821219$.

Again: in dividing $\bar{4}\cdot7689317$ by 3, we first add $\bar{2}$ to the characteristic, and $+2$ to the mantissa; the lowest multiple of 3, after 4, being 6, which exceeds 4 by 2; and $\bar{2}+2$ being $=0$. We then find that $\frac{1}{3}$ of $\bar{4}\cdot7689317$ is $\bar{2}\cdot9229772$.

$$\begin{array}{r} \bar{4}\cdot7689317 \\ \underline{\bar{2}+2} \\ 3)\overline{\bar{6}+2\cdot7689317} \\ \hline \bar{2}\cdot9229772 \end{array}$$

* This explains why the sign *minus* is always placed above (instead of before) the characteristic of the logarithm of a decimal. If placed before the characteristic, the sign would be understood to indicate that the whole of the logarithm was subtractive.

† It is obvious that if $\cdot9106095$ were added to a number, and 7 *subtracted from* the sum, the result would be exactly the same as if *3·9106095* were added to the number, and 10 subtracted from the sum.

by $\frac{1}{2+\frac{1}{3\frac{1}{5}}}$ than by $\frac{1}{2+\frac{1}{3}}$; so that, as a third approximation, and the closest yet obtained, we may set down $\frac{1}{2+\frac{1}{3\frac{1}{5}}}$, or (the terms of the fraction $\frac{1}{3\frac{1}{5}}$ being multiplied by 5) $\frac{1}{2+\frac{5}{16}}$, or (the terms of this last fraction being multiplied by 16) $\frac{16}{37}$.

The exact value of $\frac{29}{152}$ being $\frac{1}{5\frac{7}{29}}$, the original fraction can be written under the form $\frac{1}{2+\frac{1}{3+\frac{1}{5\frac{7}{29}}}}$. Dividing its terms by 7, we find that the fraction $\frac{7}{29}$ lies between $\frac{1}{4}$ and $\frac{1}{5}$, and that, consequently, it is more accurate to substitute $\frac{1}{4}$ for $\frac{7}{29}$ than to reject $\frac{7}{29}$ altogether; so that, as a fourth and still closer approximation, we may set down $\frac{1}{2+\frac{1}{3+\frac{1}{5\frac{1}{4}}}}$, or (the terms of the fraction $\frac{1}{5\frac{1}{4}}$ being multiplied by 4) $\frac{1}{2+\frac{1}{3+\frac{4}{21}}}$, or (the terms of the fraction $\frac{1}{3\frac{4}{21}}$ being multiplied by 21) $\frac{1}{2+\frac{21}{67}}$, or (the terms of this last fraction being multiplied by 67) $\frac{67}{155}$.

The exact value of $\frac{7}{29}$ being $\frac{1}{4\frac{1}{7}}$, the original fraction can be written under the form $\frac{1}{2+\frac{1}{3+\frac{1}{5+\frac{1}{4+\frac{1}{7}}}}}$. As the fraction $\frac{1}{7}$ admits of no alteration, its numerator being 1, the work of "*decomposition*"—as it is sometimes called—terminates here;

and when $\frac{1}{7}$ is taken into account, we necessarily obtain, instead of another approximation, the original fraction $\left(\frac{485}{1122}\right)$ itself:—

$$\frac{1}{4+\frac{1}{7}}=\frac{7}{29};\quad \frac{1}{5+\frac{1}{4+\frac{1}{7}}}=\frac{1}{5+\frac{7}{29}}=\frac{29}{152};\quad \frac{1}{3+\frac{1}{5+\frac{1}{4+\frac{1}{7}}}}=\frac{1}{3+\frac{29}{152}}=$$

$$\frac{152}{485};\quad \frac{1}{2+\frac{1}{3+\frac{1}{5+\frac{1}{4+\frac{1}{7}}}}}=\frac{1}{2+\frac{152}{485}}=\frac{485}{1122}.$$

Of such approximations as the foregoing,—each closer than the preceding one,—the first is too large; the second, too small; the third, too large; the fourth, too small; and so on, alternately. Thus, in writing $\frac{1}{2}$ for $\frac{485}{1122}$, which is really equivalent to $\frac{1}{2\frac{152}{485}}$, we take a fraction whose denominator is too small; so that the first approximation is too large. Again, $\frac{1}{3}$ being a larger fraction than $\frac{1}{3\frac{29}{152}}$, $2+\frac{1}{3}$ is a larger number than $2+\frac{1}{3\frac{29}{152}}$; and $\frac{1}{2+\frac{1}{3}}$ is a smaller fraction than $\frac{1}{2+\frac{1}{3\frac{29}{152}}}$, or than $\frac{485}{1122}$; so that the second approximation is too small. Moreover, $\frac{1}{5}$ being a larger fraction than $\frac{1}{5\frac{7}{29}}$, $3+\frac{1}{5}$ is a larger number than $3+\frac{1}{5\frac{7}{29}}$; $\frac{1}{3+\frac{1}{5}}$ is a smaller fraction than $\frac{1}{3+\frac{1}{5\frac{7}{29}}}$; $2+\frac{1}{3+\frac{1}{5}}$ is a smaller number than $2+\frac{1}{3+\frac{1}{5\frac{7}{29}}}$; and $\frac{1}{2+\frac{1}{3+\frac{1}{5}}}$ is a larger fraction than $\frac{1}{2+\frac{1}{3+\frac{1}{5\frac{7}{29}}}}$, or than $\frac{485}{1122}$; so that the third approximation is too large. Lastly, $\frac{1}{4}$ being a larger fraction than $\frac{1}{4\frac{1}{7}}$, $5+\frac{1}{4}$ is a larger number than $5+\frac{1}{4\frac{1}{7}}$; $\frac{1}{5+\frac{1}{4}}$

is a smaller fraction than $\dfrac{1}{5+\dfrac{1}{4\frac{1}{7}}}$; $3+\dfrac{1}{5+\frac{1}{4}}$ is a smaller number than $3+\dfrac{1}{5+\dfrac{1}{4\frac{1}{7}}}$; $\dfrac{1}{3+\dfrac{1}{5+\frac{1}{4}}}$ is a larger fraction than $\dfrac{1}{3+\dfrac{1}{5+\dfrac{1}{4\frac{1}{7}}}}$;

$2+\dfrac{1}{3+\dfrac{1}{5+\frac{1}{4}}}$ is a larger number than $2+\dfrac{1}{3+\dfrac{1}{5+\dfrac{1}{4\frac{1}{7}}}}$; and

$\dfrac{1}{2+\dfrac{1}{3+\dfrac{1}{5+\frac{1}{4}}}}$ is a smaller fraction than $\dfrac{1}{2+\dfrac{1}{3+\dfrac{1}{5+\dfrac{1}{4\frac{1}{7}}}}}$, or than $\dfrac{485}{1122}$; so that the fourth approximation is too small.

291. An expression like $\dfrac{1}{2+\dfrac{1}{3+\dfrac{1}{5+\dfrac{1}{4+\frac{1}{7}}}}}$ is called a CONTINUED FRACTION, or a CHAIN FRACTION.

292. To convert an ordinary fraction* into a Continued fraction: Divide—first, the terms of the given fraction by the numerator; next, the terms of the fractional part of the resulting denominator by *its* numerator; then, the terms of the fractional part of the new denominator by the numerator of this part; and so on, the process being continued until a fractional *unit* is obtained.

In converting $\frac{485}{1122}$ into a continued fraction, we divided—first, the terms of the given fraction by 485; next, the terms of the fraction $\frac{152}{485}$ by 152; then, the terms of the fraction $\frac{29}{152}$

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* That is, an ordinary fraction which is proper, in its simplest form, *and not a fractional unit.*

by 29; and, lastly, the terms of the fraction $\frac{7}{29}$ by 7—the work terminating when the fractional *unit* $\frac{1}{7}$ was obtained:

$$\frac{485}{1122}=\cfrac{1}{2\frac{152}{485}}=\cfrac{1}{2+\cfrac{1}{3\frac{29}{152}}}=\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{5\frac{7}{29}}}}=\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{5+\cfrac{1}{4+\cfrac{1}{7}}}}}.$$

293. When the terms of an ordinary fraction are treated as if their greatest common measure were sought, and the resulting quotients are w, x, y, z, &c., respectively, the fraction is equivalent to the continued fraction $\cfrac{1}{w+\cfrac{1}{x+\cfrac{1}{y+\cfrac{1}{z+\&c.}}}}$

This becomes evident when the work in the margin is compared with the means employed for the conversion of $\frac{485}{1122}$ into $\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{5+\cfrac{1}{4+\cfrac{1}{7}}}}}$.

```
485)1122(2
     970
     ---
     152)485(3
         456
         ---
          29)152(5
             145
             ---
               7)29(4
                 28
                 --
                  1)7(7
                    7
                    -
                    0
```

294. From the continued fraction $\cfrac{1}{w+\cfrac{1}{x+\cfrac{1}{y+\cfrac{1}{z+\&c.}}}}$

X

we obtain the following approximations, which explain themselves :—

1st, $$\frac{1}{w}$$

2nd, $$\frac{1}{w+\frac{1}{x}} = \frac{x}{w\times x+1}$$

3rd, $$\frac{1}{w+\frac{1}{x+\frac{1}{y}}} = \frac{1}{w+\frac{y}{x\times y+1}} = \frac{x\times y+1}{w\times x\times y+w+y} = \frac{x\times y+1}{(w\times x+1)\times y+w}$$

4th, $$\frac{1}{w+\frac{1}{x+\frac{1}{y+\frac{1}{z}}}} = \frac{1}{w+\frac{1}{x+\frac{z}{y\times z+1}}} = \frac{1}{w+\frac{y\times z+1}{x\times y\times z+x+z}} =$$

$$\frac{x\times y\times z+x+z}{w\times x\times y\times z+w\times x+w\times z+y\times z+1} =$$

$$\frac{(x\times y+1)\times z+x}{\left\{w\times x\times y+w+y\right\}\times z+w\times x+1} = \frac{(x\times y+1)\times z+x}{\left\{(w\times x+1)\times y+w\right\}\times z+w\times x+1}$$

&c. &c.

295. The approximations derivable from the conversion of an ordinary fraction into a continued fraction are most easily obtained as follows: Treat the terms of the given fraction as if their greatest common measure were sought, and write the resulting quotients in a horizontal row; then, set down, as the

1st approximation, $$\frac{1}{\text{1st quotient}}$$

2nd, $$\frac{\text{2nd quotient}}{\text{1st quot.}\times\text{2nd quot.}+1}$$

3rd, $$\frac{\text{numr. of 2nd approx.}\times\text{3rd quot.}+\text{numr. of 1st approx.}}{\text{denomr. of 2nd approx.}\times\text{3rd quot.}+\text{denomr. of 1st approx.}}$$

4th, $$\frac{\text{numr. of 3rd approx.}\times\text{4th quot.}+\text{numr. of 2nd approx.}}{\text{denomr. of 3rd approx.}\times\text{4th quot.}+\text{denomr. of 2nd approx.}}$$

5th, $$\frac{\text{numr. of 4th approx.}\times\text{5th quot.}+\text{numr. of 3rd approx.}}{\text{denomr. of 4th approx.}\times\text{5th quot.}+\text{denomr. of 3rd approx.}}$$

&c. &c.

This follows from § 294:

(I.)	(II.)	(III.)	(IV.)
		Quotients.	
w	x	y	z
		Approximations.	
$\frac{1}{w}$	$\frac{x}{w\times x+1}$	$\frac{x\times y+1}{(w\times x+1)\times y+w}$	$\frac{(x\times y+1)\times z+x}{\{(w\times x+1)\times y+w\}\times z+w\times x+1}$

In the case of the fraction $\frac{485}{1122}$, the "quotients," as we have seen, are 2, 3, 5, 4, and 7. Dividing unity by the first of these quotients, we obtain the first approximation, $\frac{1}{2}$. Adding unity to the product of the first two quotients, and dividing the second quotient by the result, we obtain the second approximation, $\left(\frac{3}{2\times3+1}=\right)\frac{3}{7}$. Multiplying the terms of the second approximation by the third quotient, and adding to the products the terms of the first approximation, respectively, we obtain the third approximation, $\left(\frac{3\times5+1}{7\times5+2}=\right)\frac{16}{37}$. Multiplying the terms of the third approximation by the fourth quotient, and adding to the products the terms of the second approximation, respectively, we obtain the fourth approximation, $\left(\frac{16\times4+3}{37\times4+7}=\right)\frac{67}{155}$. Multiplying the terms of the fourth approximation by the fifth quotient, and adding to the products the terms of the third approximation, respectively, we obtain, not a new "approximation"—the fifth quotient being the last, but $\left(\frac{67\times7+16}{155\times7+37}=\right)\frac{485}{1122}$, the original fraction itself:

	(I.)	(II.)	(III.)	(IV.)	(V.)
Quotients	2	3	5	4	7
Approximations	$\frac{1}{2}$	$\frac{3}{7}$	$\frac{16}{37}$	$\frac{67}{155}$	$\frac{485}{1122}$

Of such a series of approximations as this, every two consecutive terms will be found to differ by a fractional unit whose denominator is the product of the denominators of the two terms. Thus, $\frac{1}{2}$ and $\frac{3}{7}$ differ by $\frac{1}{2\times7}$; $\frac{3}{7}$ and $\frac{16}{37}$, by $\frac{1}{7\times37}$; $\frac{16}{37}$ and $\frac{67}{155}$, by $\frac{1}{37\times155}$; and $\frac{67}{155}$ and $\frac{485}{1122}$, by $\frac{1}{155\times1122}$. So that $\frac{485}{1122}$, which lies between every two consecutive approxi-

mations, differs from

$$\left.\begin{array}{c}\frac{1}{2}\\[4pt]\frac{3}{7}\\[4pt]\frac{16}{37}\\[4pt]\frac{67}{155}\end{array}\right\}\text{ by less than }\left\{\begin{array}{c}\frac{1}{14}\\[4pt]\frac{1}{259}\\[4pt]\frac{1}{5735}\\[4pt]\frac{1}{173910}\end{array}\right.$$

Let us next take $\frac{203519}{645044}$. Treating the terms of this fraction in the manner shown below, we obtain the quotients

3, 5, 1, 9, 7, 2, 4, 6, and 8:

```
203519)645044(3
       610557
       ------
        34487)203519(5
              172435
              ------
               31084)34487(1
                     31084
                     -----
                      3403)31084(9
                           30627
                           -----
                             457(3403(7
                                 3199
                                 ----
                                  204)457(2
                                      408
                                      ---
                                       49)204(4
                                          196
                                          ---
                                            8)49(6
                                              48
                                              --
                                               1)8(8
                                                 8
                                                 -
                                                 0
```

From these quotients we obtain the following approximations to $\frac{203519}{645044}$:

1st approximation, $\frac{1}{3}$

2nd approximation,		$\dfrac{5}{3\times5+1}=\dfrac{5}{16}$
3rd	,,	$\dfrac{5\times1+1}{16\times1+3}=\dfrac{6}{19}$
4th	,,	$\dfrac{6\times9+5}{19\times9+16}=\dfrac{59}{187}$
5th	,,	$\dfrac{59\times7+6}{187\times7+19}=\dfrac{419}{1328}$
6th	,,	$\dfrac{419\times2+59}{1328\times2+187}=\dfrac{897}{2843}$
7th	,,	$\dfrac{897\times4+419}{2843\times4+1328}=\dfrac{4007}{12700}$
8th	,,	$\dfrac{4007\times6+897}{12700\times6+2843}=\dfrac{24939}{79043}$

Original fraction, $\dfrac{24939\times8+4007}{79043\times8+12700}=\dfrac{203519}{645044}$

In practice, the work would stand thus:

3	5	1	9	7	2	4	6	8
$\frac{1}{3}$	$\frac{5}{16}$	$\frac{6}{19}$	$\frac{59}{187}$	$\frac{419}{1328}$	$\frac{897}{2843}$	$\frac{4007}{12700}$	$\frac{24939}{79043}$	$\frac{203519}{645044}$

And the contd. fraction would be $\cfrac{1}{3+\cfrac{1}{5+\cfrac{1}{1+\cfrac{1}{9+\cfrac{1}{7+\cfrac{1}{2+\cfrac{1}{4+\cfrac{1}{6+\cfrac{1}{8}}}}}}}}}$

$$\frac{203519}{645044}=\frac{1}{3\frac{34487}{203519}}=\cfrac{1}{3+\cfrac{1}{5\frac{31084}{34487}}}=\cfrac{1}{3+\cfrac{1}{5+\cfrac{1}{1\frac{3403}{31084}}}}=\cfrac{1}{3+\cfrac{1}{5+\cfrac{1}{1+\cfrac{1}{9\frac{457}{3403}}}}}=$$

$$\cfrac{1}{3+\cfrac{1}{5+\cfrac{1}{1+\cfrac{1}{9+\cfrac{1}{7\frac{204}{457}}}}}}=\cfrac{1}{3+\cfrac{1}{5+\cfrac{1}{1+\cfrac{1}{9+\cfrac{1}{7+\cfrac{1}{2\frac{49}{204}}}}}}}=\cfrac{1}{3+\cfrac{1}{5+\cfrac{1}{1+\cfrac{1}{9+\cfrac{1}{7+\cfrac{1}{2+\cfrac{1}{4\frac{8}{49}}}}}}}}=\cfrac{1}{3+\cfrac{1}{5+\cfrac{1}{1+\cfrac{1}{9+\cfrac{1}{7+\cfrac{1}{2+\cfrac{1}{4+\cfrac{1}{6+\frac{1}{8}}}}}}}}}$$

As a third and practical illustration, let it be required to express, in terms of a day, the difference between the solar year (365 d. 5 h. 48 min. 49 sec.) and 365 days. The number of seconds in the difference is 20,929, and there are 86,400 seconds in 24 hours. In terms of a day, therefore, the difference is expressed by the fraction $\frac{20929}{86400}$. Dealing with this fraction in the manner already explained, we obtain the following results:

Quotients,	4	7	1	3	1	16	1	1	15
Approximations,	$\frac{1}{4}$	$\frac{7}{29}$	$\frac{8}{33}$	$\frac{31}{128}$	$\frac{39}{161}$	$\frac{655}{2704}$	$\frac{694}{2865}$	$\frac{1349}{5569}$	$\frac{20929}{86400}$

So that the difference between the solar year and 365 days amounts to nearly 1 day in 4 years, to a little more than 7 days in 29 years, to somewhat less than 8 days in 33 years,* to very little more than 39 days in 161 years, &c.

Continued fractions suggest a means—different from that already explained—for the extraction of square roots. Let it be required, for example, to extract $\sqrt{17}$. This root lies between 4 and 5. Regarding 4 as the "known" part of the root, and putting u for the "unknown" part, we have $4+u=\sqrt{17}$; $(4+u)^2=17$; $4^2+2\times4\times u+u^2=17$; $16+8\times u+u^2=17$; $8\times u+u^2=(17-16=)1$; $(8+u)\times u=1$; $u=\frac{1}{8+u}=\cfrac{1}{8+\cfrac{1}{8+u}}=$

$$\cfrac{1}{8+\cfrac{1}{8+\cfrac{1}{8+u}}}=\cfrac{1}{8+\cfrac{1}{8+\cfrac{1}{8+\cfrac{1}{8+u}}}}=\cfrac{1}{8+\cfrac{1}{8+\cfrac{1}{8+\cfrac{1}{8+\cfrac{1}{8+\&c.}}}}}\dagger:$$

* It is an interesting fact that the addition of 8 days to every 33 years (of 365 days each) was proposed by the Persian astronomers nearly seven centuries ago.

† *Each* of these fractions is formed from the preceding one by the *substitution* of $\frac{1}{8+u}$ for u.

8	8	8	8	8	&c.
$\frac{1}{8}$	$\frac{8}{65}$	$\frac{65}{528}$	$\frac{528}{4289}$	$\frac{4289}{34840}$	&c.

Here we have an interminate series of approximations—the continued fraction being interminate. Without, however, proceeding farther than the third approximation, we obtain a sufficiently accurate result: $u=\frac{65}{528}=\cdot 123$; $\sqrt{17}=4+u=4\cdot 123$.

Next, let it be required to extract $\sqrt{41}$. This root lies between 6 and 7. Regarding 6 as the known part of the root, and putting x for the unknown part, we have $6+x=\sqrt{41}$; $(6+x)^2=41$; $6^2+2\times 6\times x+x^2=41$; $36+12\times x+x^2=41$; $12\times x+x^2=(41-36=)\ 5$; $(12+x)\times x=5$; $x=\frac{5}{12+x}=$

$$\frac{5}{12+\frac{5}{12+x}}=\frac{5}{12+\frac{5}{12+\frac{5}{12+x}}}=\frac{5}{12+\frac{5}{12+\frac{5}{12+\frac{5}{12+x}}}}=\frac{5}{12+\frac{5}{12+\frac{5}{12+\frac{5}{12+\frac{5}{12+\&c.}}}}}*$$

Taking, as a sufficiently close approximation, $\frac{5}{12+\frac{5}{12+\frac{5}{12}}}$ and converting this expression into an ordinary fraction, we thus find $x=\frac{745}{1848}=\cdot 403$; $\sqrt{41}=6+x=6\cdot 403$:

$$\frac{5}{12+\frac{5}{12+\frac{5}{12}}}=\frac{5}{12+\frac{60}{149}}=\frac{745}{1848}.$$

Continued fractions enable us to express any number as a power of any other number. Let us suppose the fact to have

---

* Each of these fractions is formed from the preceding one by the *substitution of* $\frac{5}{12+x}$ for x.

been, in some way, ascertained that 409 is the 292nd power of the 87th root of 6:

$$409=6^{\frac{292}{87}}.$$

Putting a for the 87th root of 6, we have

$$409=a^{292}$$
$$6=a^{87}$$

Dividing 409 by 6 as often as possible,

$$\frac{409}{6} \text{ or } 68\cdot1\dot{6}=\frac{a^{292}}{a^{87}}=a^{205}$$

$$\frac{68\cdot1\dot{6}}{6} \text{ or } 11\cdot36\dot{1}=\frac{a^{205}}{a^{87}}=a^{118}$$

$$\frac{11\cdot36\dot{1}}{6} \text{ or } 1\cdot8935185=\frac{a^{118}}{a^{87}}=a^{31}$$

It thus appears that, as often as 6 can be divided into 409, so often, exactly, can 87 be subtracted from 292.

Substituting $6^{\frac{1}{87}}$ for a, we have

$$1\cdot8935185 = 6^{\frac{31}{87}}$$
$$1\cdot8935185^{87}= 6^{31}$$
$$1\cdot8935185^{\frac{87}{31}}=6$$

Putting b for the 31st root of $1\cdot8935185$,

$$6 =b^{87}$$
$$1\cdot8935185=b^{31}$$

Dividing 6 by $1\cdot8935185$ as often as possible,

$$\frac{6}{1\cdot8935185} \text{ or } 3\cdot1687042=\frac{b^{87}}{b^{31}}=b^{56}$$

$$\frac{3\cdot1687042}{1\cdot8935185} \text{ or } 1\cdot6734477=\frac{b^{56}}{b^{31}}=b^{25}$$

It thus appears that $1\cdot8935185$ can be divided into 6 as often—and *only* as often—as 31 can be subtracted from 87.

Substituting $1\cdot8935185^{\frac{1}{31}}$ for b, we have

$$1\cdot6734477 =1\cdot8935185^{\frac{25}{31}}$$
$$1\cdot6734477^{31}=1\cdot8935185^{25}$$
$$1\cdot6734477^{\frac{31}{25}}=1\cdot8935185$$

Putting c for the 25th root of $1\cdot6734477$,

$$1\cdot8935185=c^{31}$$
$$1\cdot6734477=c^{25}$$

$$\frac{1\cdot8935185}{1\cdot6734477} \text{ or } 1\cdot1315074=\frac{c^{31}}{c^{25}}=c^{6}$$

It thus appears that $1\cdot6734477$ can be divided once only into *1·8935185*, and that 25 can be subtracted once only from 31.

Substituting $1\cdot6734477^{\frac{1}{25}}$ for c, we have

$$1\cdot1315074 = 1\cdot6734477^{\frac{6}{25}}$$
$$1\cdot1315074^{25} = 1\cdot6734477^{6}$$
$$1\cdot1315074^{\frac{25}{6}} = 1\cdot6734477$$

Putting d for the 6th root of $1\cdot1315074$,

$$1\cdot6734477 = d^{25}$$
$$1\cdot1315074 = d^{6}$$

Dividing $1\cdot6734477$ by $1\cdot1315074$ as often as possible,

$$\frac{1\cdot6734477}{1\cdot1315074} \text{ or } 1\cdot4789543 = \frac{d^{25}}{d^{6}} = d^{19}$$
$$\frac{1\cdot4789543}{1\cdot1315074} \text{ or } 1\cdot3070655 = \frac{d^{19}}{d^{6}} = d^{13}$$
$$\frac{1\cdot3070655}{1\cdot1315074} \text{ or } 1\cdot1551542 = \frac{d^{13}}{d^{6}} = d^{7}$$
$$\frac{1\cdot1551542}{1\cdot1315074} \text{ or } 1\cdot0208985 = \frac{d^{7}}{d^{6}} = d.$$

It thus appears that $1\cdot1315074$ can be divided into $1\cdot6734477$ as often—and *only* as often—as 6 can be subtracted from 25.

Substituting $1\cdot1315074^{\frac{1}{6}}$ for d, we have

$$1\cdot0208985 = 1\cdot1315074^{\frac{1}{6}}$$
$$1\cdot0208985^{6} = 1\cdot1315074$$
$$\text{or } 1\cdot0208985^{\frac{6}{1}} = 1\cdot1315074.$$

Now, as often as 1 can be subtracted from 6, so often, exactly, ought $1\cdot0208985$ to be contained as factor in $1\cdot1315074$; and this we find to be the case:

$$\frac{1\cdot1315074}{1\cdot0208985} = 1\cdot1083447$$
$$\frac{1\cdot1083447}{1\cdot0208985} = 1\cdot0856561$$
$$\frac{1\cdot0856561}{1\cdot0208985} = 1\cdot0634319$$
$$\frac{1\cdot0634319}{1\cdot0208985} = 1\cdot0416627$$
$$\frac{1\cdot0416627}{1\cdot0208985} = 1\cdot0203391$$
$$\frac{1\cdot0203391}{1\cdot0208985} = 1 \text{ very nearly.*}$$

* This last quotient would be *exactly* 1 if 409 were exactly the 292nd power of the 87th root of 6; but the index $\frac{292}{87}$ is only an approximation.

Unaccompanied by explanation, the results of the preceding divisions and subtractions may be written in this way:—

$6)409(68\cdot1\dot{6}$; $11\cdot3\dot{6}\dot{1}$; $1\cdot8935185$ [3 divisions.]
$1\cdot8935185)6(3\cdot1687042$; $1\cdot6734477$ [2 divisions.]
$1\cdot6734477)1\cdot8935185(1\cdot1315074$ [1 division.]
$1\cdot1315074)1\cdot6734477(1\cdot4789543$; $1\cdot3070655$; $1\cdot1551542$; $1\cdot0208985$ [4 divisions.]
$1\cdot0208985)1\cdot1315074(1\cdot1083447$; $1\cdot0856561$; $1\cdot0634319$; $1\cdot0416627$; $1\cdot0203391$; 1 [6 divisions.]

$292-87=205$; $205-87=118$; $118-87=31$ [3 subtractions.]
$87-31=56$; $56-31=25$ [2 subtractions.]
$31-25=6$ [1 subtraction.]
$25-6=19$; $19-6=13$; $13-6=7$; $7-6=1$ [4 subtractions.]
$6-1=5$; $5-1=4$; $4-1=3$; $3-1=2$; $2-1=1$; $1-1=0$ [6 subtractions.]

It will be seen that, in working this series of subtractions, we proceed exactly as if, without the aid of division, we desired to find the greatest common measure of 292 and 87—subtracting 87 as often as possible from 292 being the same as dividing 292 by 87; subtracting 31 as often as possible from 87, the same as dividing 87 by 31; and so on:

$$\frac{292}{87}=3\tfrac{31}{87}=3+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{4+\frac{1}{6}}}}$$

```
87)292(3
   261
   ---
    31)87(2
       62
       --
       25)31(1
          25
          --
           6)25(4
             24
             --
              1)6(6
                6
                -
                0
```

By means of the "quotients" 3, 2, 1, 4, 6, we can at once *determine* $\frac{292}{87}$, the index of the power which 409 is of 6; and *we find these* quotients by treating 409 and 6 in the manner

just explained, and noting the number of divisions (3, 2, 1, 4, and 6, respectively), performed with each divisor: 6 being first divided as often as possible into 409; 1·8935185, the last of the resulting quotients, being next divided as often as possible into 6; 1·6734477, the last of the second set of quotients, being then divided as often as possible into 1·8935185; and so on—the process being continued until 1 is obtained for quotient:

3	2	1	4	6
	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{5}{14}$	$\frac{31}{87}$

Knowing that the index of the power which 409 is of 6 must be greater than unity, we set down 3, the first "quotient," as the integral part of the mixed number to which the index is reducible: we then find, in the usual way, that the fractional part is $\frac{31}{87}$, and that, consequently, the index is $3\frac{31}{87}$, or $\frac{292}{87}$. So that, if the base were 6, the logarithm of 409 would be 3·3563218.

Next, let it be required to find the "Common" logarithm of 5: in other words, to express 5 as a power of 10. Proceeding as in the last case, we obtain the following results:—

5)10(2. [1 division.]
2)5(2·5; 1·25. [2 divisions.]
1·25)2(1·6; 1·28; 1·024. [3 divisions.]
1·024)1·25(1·2207031; 1·1920929; 1·1641532; 1·1368684; 1·1102230; 1·0842021; 1·0587911; 1·0339757; 1·0097419. [9 divisions.]
1·0097419)1·924(1·0141205; 1·0043363. [2 divisions]
1·0043363)1·0097419(1·0053822; 1·0010414. [2 divisions.]
1·0010414)1·0043363(1·0032915; 1·0022477; 1·0012050; 1·0001634. [4 divisions.]
1·0001634)1·0010414(1·0008778; 1·0007143; 1·0005508; 1·0003873; 1·0002238; 1·0000604. [6 divisions.]
1·0000604)1·0001634(1·0001030; 1·0000426. [2 divisions.]
1·0000426)1·0000604(1·0000178. [1 division.]
1·0000178)1·0000426(1·0000248; 1·0000070. [2 divisions.]
1·0000070)1·0000178(1·0000108; 1·0000038. [2 divisions.]
1·0000038)1·0000070(1·0000032. [1 division.]
1·0000032)1·0000038(1·0000006. [1 division.]

The difference between this last quotient and 1 being almost inappreciable, the divisions terminate here; and the index of the power which 5 is of 10—in other words, the logarithm of 5—*is found, as follows*, to be $\frac{453043}{648158}$, or ·6989700:—

1	2	3	9	2	2	4	6	2	1	2	2	1	1
$\frac{1}{1}$	$\frac{2}{3}$	$\frac{7}{10}$	$\frac{65}{93}$	$\frac{137}{196}$	$\frac{339}{485}$	$\frac{1493}{2136}$	$\frac{9297}{13301}$	$\frac{20087}{28738}$	$\frac{29384}{42039}$	$\frac{78855}{112816}$	$\frac{187094}{267671}$	$\frac{265949}{380487}$	$\frac{453043}{648158}$

Putting $\frac{x}{y}$ for the required index, we have—

```
x)y(1
  x
  —
  a)x(2
    2a
    —
    b)a(3
      3b
      —
      c)b(9
        9c
        —
        d)c(2
          2d
          —
          e)d(2
            2e
            —
            f)e(4
              4f
              —
              ..

g)f(6
  6g
  —
  h)g(2
    2h
    —
    k)h(1
      k
      —
      l)k(2
        2l
        —
        m)l(2
          2m
          —
          n)m(1
            n
            —
            p)n(1
              p
              —
              o
```

$$\frac{x}{y}=\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{9+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{4+\cfrac{1}{6+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{1+\frac{1}{1}}}}}}}}}}}}}}$$

$$=\frac{453043}{648158}=\cdot6989700$$

COMPOUND INTEREST.

296. Compound interest—which is commonly, but erroneously, defined to be "interest on interest" —may be regarded as consisting of two parts, namely: (*a*) *interest on principal*, and (*b*) *interest on interest.*

If £1,000 were lent for 3 years, at 5 per cent., and the interest paid in yearly instalments, the lender would receive, as interest on principal,—or as simple interest,—at the end of the

1st year, ...	£50
2nd „	£50
3rd „	£50
Altogether, . .	£150

In addition to this, the lender would receive £7 12*s.* 6*d.*—interest on interest—by investing, as principal (at 5 per cent.), the first £50 for the last two years, the second £50 for the third year, and a further sum of £2 10*s.* for the third year; *i.e.*, the £2 10*s.* falling due on the first £50 at the end of the second year. So that the compound interest on £1,000 for 3 years, at 5 per cent., would be (£150+£7 12*s.* 6*d.*=) £157 12*s.* 6*d.*:

					£	s.	d.
Interest on	1st £50	for	2nd year	=	2	10	0
			3rd „	=	2	10	0
	2nd £50	„	3rd „	=	2	10	0
	1st £2 10*s.*	„	„ „	=	0	2	6
Total interest on			interest	=	7	12	6
			principal	=	150	0	0
Compound interest				=	157	12	6

This result can be obtained in a different way. Remembering that the "amount" (*i.e.*, the principal+the interest) for any particular year is the "principal" available for the following year, we have—

£	s.	d.		
1,000	0	0	= 1st year's	principal
50	0	0		interest
1,050	0	0		amount

£	s.	d.		
1,050	0	0	=2nd year's	principal
52	10	0		interest
1,102	10	0		amount
1,102	10	0	=3rd year's	principal
55	2	6		interest
1,157	12	6		amount
1,000	0	0	=original principal	
157	12	6	=compound interest.	

It is hardly necessary to observe that the calculation of compound interest in either of the preceding ways would be extremely tedious and troublesome—particularly if the time for which the principal was lent extended over a great many years. But compound interest can be calculated more easily.

If £1 were invested, as principal, at (say) 5 per cent., on the understanding that the "interest on principal" should fall due in annual instalments, the first year's interest would be £·05, and the first year's amount (£1+£·05=) £1·05. This amount would become the second year's principal; so that—two principals being proportional to their respective amounts for *equal* periods of time, when the rate per cent. is the same in both cases—the second year's amount would be the last term of the proportion

(1st year's prin.)		(1st year's amt.)		(2nd year's prin.)		(2nd year's amt.)
£1	:	£1·05	::	£1·05	:	$£1·05^2$

As $£1·05^2$, the second year's amount, would become the third year's principal, the third year's amount would be the last term of the proportion

(1st year's prin.)		(1st year's amt.)		(3rd year's prin.)		(3rd year's amt.)
£1	:	£1·05	::	$£1·05^2$	:	$£1·05^3$

As $£1·05^3$, the third year's amount, would become the fourth year's principal, the fourth year's amount would be the last term of the proportion

(1st year's prin.)		(1st year's amt.)		(4th year's prin.)		(4th year's amt.)
£1	:	£1·05	::	$£1·05^3$	:	$£1·05^4$

It could be shown, in the same manner, that the fifth year's

amount would be £$1{\cdot}05^5$; the sixth year's, £$1{\cdot}05^6$; the seventh year's, £$1{\cdot}05^7$; &c.:—

$$\left.\begin{array}{l}\pounds 1\\ \pounds 1{\cdot}05\end{array}\right\}=\text{1st year's}\left\{\begin{array}{l}\text{principal}\\ \textit{amount}\end{array}\right.$$
$$\left.\begin{array}{l}\pounds 1{\cdot}05\\ \pounds 1{\cdot}05^2\end{array}\right\}=\text{2nd year's}\left\{\begin{array}{l}\text{principal}\\ \textit{amount}\end{array}\right.$$
$$\left.\begin{array}{l}\pounds 1{\cdot}05^2\\ \pounds 1{\cdot}05^3\end{array}\right\}=\text{3rd year's}\left\{\begin{array}{l}\text{principal}\\ \textit{amount}\end{array}\right.$$
$$\left.\begin{array}{l}\pounds 1{\cdot}05^3\\ \pounds 1{\cdot}05^4\end{array}\right\}=\text{4th year's}\left\{\begin{array}{l}\text{principal}\\ \textit{amount}\end{array}\right.$$
&c. &c.

If the interest on principal fell due in half-yearly instalments, the first half-year's interest would be £$\cdot 025$, and the first half-year's amount (£$1+$£$\cdot 025=$)£$1{\cdot}025$. This amount would become the second half-year's principal. So that, from a series of proportions like the preceding, the second half-year's amount would be found to be £$1{\cdot}025^2$; the third half-year's, £$1{\cdot}025^3$; the fourth half-year's, £$1{\cdot}025^4$; &c.:—

$$\left.\begin{array}{l}\pounds 1\\ \pounds 1{\cdot}025\end{array}\right\}=\text{1st half-year's}\left\{\begin{array}{l}\text{principal}\\ \textit{amount}\end{array}\right.$$
$$\left.\begin{array}{l}\pounds 1{\cdot}025\\ \pounds 1{\cdot}025^2\end{array}\right\}=\text{2nd half-year's}\left\{\begin{array}{l}\text{principal}\\ \textit{amount}\end{array}\right.$$
$$\left.\begin{array}{l}\pounds 1{\cdot}025^2\\ \pounds 1{\cdot}025^3\end{array}\right\}=\text{3rd half-year's}\left\{\begin{array}{l}\text{principal}\\ \textit{amount}\end{array}\right.$$
$$\left.\begin{array}{l}\pounds 1{\cdot}025^3\\ \pounds 1{\cdot}025^4\end{array}\right\}=\text{4th half-year's}\left\{\begin{array}{l}\text{principal}\\ \textit{amount}\end{array}\right.$$
&c. &c.

If the interest on principal fell due in quarterly instalments, the first quarter's interest would be £$\cdot 0125$, and the first quarter's amount (£$1+$£$\cdot 0125=$)£$1{\cdot}0125$. This amount would become the second quarter's principal. So that the second quarter's amount would be £$1{\cdot}0125^2$; the third quarter's, £$1{\cdot}0125^3$; the fourth quarter's, £$1{\cdot}0125^4$; &c.:—

$$\left.\begin{array}{l}\pounds 1\\ \pounds 1{\cdot}0125\end{array}\right\}=\text{1st quarter's}\left\{\begin{array}{l}\text{principal}\\ \textit{amount}\end{array}\right.$$
$$\left.\begin{array}{l}\pounds 1{\cdot}0125\\ \pounds 1{\cdot}0125^2\end{array}\right\}=\text{2nd quarter's}\left\{\begin{array}{l}\text{principal}\\ \textit{amount}\end{array}\right.$$
$$\left.\begin{array}{l}\pounds 1{\cdot}0125^2\\ \pounds 1{\cdot}0125^3\end{array}\right\}=\text{3rd quarter's}\left\{\begin{array}{l}\text{principal}\\ \textit{amount}\end{array}\right.$$
$$\left.\begin{array}{l}\pounds 1{\cdot}0125^3\\ \pounds 1{\cdot}0125^4\end{array}\right\}=\text{4th quarter's}\left\{\begin{array}{l}\text{principal}\\ \textit{amount}\end{array}\right.$$
&c. &c.

297. To determine the amount of £1, at compound interest, when the time and rate are known: Find the amount of £1 for the first "period"—*i.e.*, for the first year, half-year, or quarter, according as the interest on principal falls due in yearly, half-yearly, or quarterly instalments; and raise this amount to the power indicated by the number of periods.

We have just seen that the amount of £1 for 3 years, at 5 per cent., compound interest, would be

$$\left.\begin{array}{l}£1{\cdot}05^3\\ £1{\cdot}025^6\\ £1{\cdot}0125^{12}\end{array}\right\}\text{ if the period were a }\left\{\begin{array}{l}\text{year.}\\ \text{half-year.}\\ \text{quarter.}\end{array}\right.$$

It is obvious that—all other circumstances being the same—when the amount of £1 is multiplied by

$$\left.\begin{array}{c}2\\ 3\\ 4\end{array}\right\}\text{ we obtain the amount of }\left\{\begin{array}{l}£2\\ £3\\ £4\end{array}\right.$$
&c. &c.

298. The principal, time, and rate being known, to determine the compound interest: Find the amount of £1 for the given time, and at the given rate (§ 297); multiply this amount by the principal; and subtract the principal from the resulting product.

Example I.—What compound interest would £1,000 have produced in 3 years, at 5 per cent.—interest on principal falling due in annual instalments?

Amount of £1=£1·05³=£1·157625; amount of £1,000=£1·157625 × 1,000=£1,157·625=£1,157 12*s.* 6*d.*; required compound interest=£157 12*s.* 6*d.*—the difference between £1,157 12*s.* 6*d.* and £1,000.

Note 1.—If interest on principal fell due in half-yearly instalments, we should have—

Amount of £1=£1·025⁶=£1·1597 (nearly); amount of £1,000=£1·1597 × 1,000=£1,159·7=£1,159 14*s.*; required compound interest = £159 14*s.*—the difference between £1,159 14*s.* and £1,000.

And if interest on principal fell due in quarterly instalments, we should have—

Amount of £1=£1·0125¹²=£1·16075; amount of £1,000=

£1·16075 × 1,000 = £1,160·75 = £1,160 15*s.*; required compound interest = £160 15*s.* — the difference between £1,160 15*s.* and £1,000.

Note 2.—The amount of £1 for the first period being represented by a, and the number of periods by n, the amount of £1 for n periods would be a^n (§ 297); so that, for n periods, the amount of

$$\left.\begin{array}{l} £2 \\ £3 \\ £4 \end{array}\right\} \text{would be} \left\{\begin{array}{l} £a^n \times 2 \\ £a^n \times 3 \\ £a^n \times 4 \end{array}\right.$$
&c. &c.

If, therefore, we put P for the principal, and A for the sum to which P would have amounted in n periods, we shall have

$$a^n \times P = A.$$

This easily-remembered formula enables us to deal with any exercise in Compound Interest.

299. The amount, time, and rate being known, to determine the principal: Find what the amount would be if the principal were £1 (§ 297); and divide the result into the given amount.

Example II.—What sum would have amounted, in 3 years, at 5 per cent., to £1,157 12*s.* 6*d.*—interest on principal falling due in annual instalments?

Here we have a (the amount of £1 for the first period) = £1·05; n (the number of periods) = 3; a^n (the amount of £1 for n periods) = £$1·05^3$; A (the given amount) = £1,157·625; £$1·05^3$ × P = £1,157·625; P (the required principal) =

$$£\frac{1,157·625}{1·05^3} = £\frac{1,157·625}{1·157625} = £1,000.$$

Note.—If interest on principal fell due in half-yearly instalments, we should have a = £1·025; n = 6; a^n = £$1·025^6$; £$1·025^6$ × P = £1,157·625; $P = £\frac{1,157·625}{1·025^6} = £\frac{1,157·625}{1·1597} =$ £998 4*s.* 3*d.* (nearly.)

And if interest on principal fell due in quarterly instalments, we should have a = £1·0125; n = 12; a^n = £$1·0125^{12}$; £$1·0125^{12}$ × P = £1,157·625; $P = £\frac{1,157·625}{1·0125^{12}} = £\frac{1,157·625}{1·16075} =$ £997 6*s.* 2*d.* (nearly.)

300. The principal, amount, and rate being known, to determine the time: Divide the amount by the *principal*, and find what power the resulting quo-

tient is of the amount of £1 for the first period; the index of this power will be the number of periods in the required time.

EXAMPLE III.—In what time would £1,000 have amounted to £1,157 12*s.* 6*d.*, at 5 per cent.—interest on principal falling due in annual instalments?

Here we have a=£1·05; P=£1,000; A=£1,157·625; £$1{\cdot}05^n \times 1{,}000$=£1,157·625; $1{\cdot}05^n$=£1,157·625 ÷ 1,000= 1·157625; n=3—the number of times 1·05 is contained, as factor, in 1·157625. So that, the period being a year, the required time is 3 years.

EXAMPLE IV.—In what time would £1,000 have amounted to £1,159 14*s.*, at 5 per cent.—interest on principal falling due in half-yearly instalments?

Here we have a=£1·025; P=£1,000; A=£1,159·7; £$1{\cdot}025^n \times 1{,}000$=£1,159·7; $1{\cdot}025^n$=1,159·7÷1,000=1·1597; n=6—the number of times 1·025 is contained, as factor, in 1·1597. So that, the period being a half-year, the required time is 6 half-years, or 3 years.

EXAMPLE V.—In what time would £1,000 have amounted to £1,160 15*s.*, at 5 per cent.—interest on principal falling due in quarterly instalments?

Here we have a=£1·0125; P=£1,000; A=£1,160·75; £$1{\cdot}0125^n \times 1{,}000$=£1,160·75; $1{\cdot}0125^n$=1,160·75÷1,000= 1·16075; n=12—the number of times 1·0125 is contained, as factor, in 1·16075. So that, the period being a quarter, the required time is 12 quarters, or 3 years.

301. The principal, amount, and time being known, to determine the rate: Divide the amount by the principal; from the resulting quotient evolve the root indicated by the number of periods in the given time; subtract 1 from this root; and multiply the remainder by 100. The product so obtained will be the rate, or half the rate, or a fourth of the rate—according as the period is a year, a half-year, or a quarter.

EXAMPLE VI.—At what rate would £1,000 have produced £1,157 12*s.* 6*d.* in 3 years—interest on principal falling due in annual instalments?

Here we have n=3; P=£1,000; A=£1,157·625; $a^3 \times 1{,}000 = 1{,}157{\cdot}625$; $a^3 = 1{,}157{\cdot}625 \div 1{,}000 = 1{\cdot}157625$; $a = \sqrt[3]{1{\cdot}157625} = 1{\cdot}05$, the amount of £1 for one period—i.e., for

a year; $1\cdot05-1=\cdot05$, the interest of £1 for a year; $\cdot05\times100=5$, the interest of £100 for a year—*i.e.*, the required rate per cent.

Example VII.—At what rate would £1,000 have amounted to £1,159 14*s*. in 3 years—interest on principal falling due in half-yearly instalments?

Here we have $n=6$; $P=£1,000$; $A=£1,159\cdot7$; $a^6\times1,000=1,159\cdot7$; $a^6=1,159\cdot7\div1,000=1\cdot1597$; $a=\sqrt[6]{1\cdot1597}=1\cdot025$, the amount of £1 for one period—*i.e.*, for a half-year; $1\cdot025-1=\cdot025$, the interest of £1 for a half-year; $\cdot025\times100=2\cdot5$, the interest of £100 for a half-year; $2\cdot5\times2=5$, the interest of £100 for a year—*i.e.*, the required rate per cent.

Example VIII.—At what rate would £1,000 have amounted to £1,160 15*s*. in 3 years—interest on principal falling due in quarterly instalments?

Here we have $n=12$; $P=£1,000$; $A=£1,160\cdot75$; $a^{12}\times1,000=1,160\cdot75$; $a^{12}=1,160\cdot75\div1,000=1\cdot16075$; $a=\sqrt[12]{1\cdot16075}=1\cdot0125$, the amount of £1 for one period—*i.e.*, for a quarter; $1\cdot0125-1=\cdot0125$, the interest of £1 for a quarter; $\cdot0125\times100=1\cdot25$, the interest of £100 for a quarter; $1\cdot25\times4=5$, the interest of £100 for a year—*i.e.*, the required rate per cent.

Note.—To find the number of periods in which any principal would have doubled itself at compound interest, we divide the logarithm of 2 by the logarithm of the amount of £1 for the first period. Because, substituting 2 P for A in the formula $a^n\times P=A$, we have $a^n\times P=2P$; $a^n=2$; $n=\frac{\log 2}{\log a}$. So that, if the rate were 5 per cent., money would, at compound interest, have (*a*) nearly doubled itself in 14 years, (*b*) very nearly doubled itself in 28 half-years, and (*c*) more than doubled itself in 56 quarters—according as interest on principal fell due in yearly, half-yearly, or quarterly instalments:

$$(a.)\ \frac{\log 2}{\log 1\cdot05}=\frac{\cdot3010300}{\cdot0211893}=14\cdot2\ \text{(years)}.$$

$$(b.)\ \frac{\log 2}{\log 1\cdot025}=\frac{\cdot3010300}{\cdot0107239}=28\cdot07\ \text{(half-years)}.$$

$$(c.)\ \frac{\log 2}{\log 1\cdot0125}=\frac{\cdot3010300}{\cdot0053950}=55\cdot8\ \text{(quarters)}.$$

It could be shown, in the same way, that the number of periods in which money would have trebled itself at compound interest is the quotient obtained when the logarithm of 3 is divided by the logarithm of the amount of £1 for the first period.

ANNUITIES.

302. Property of any description, when given as an equivalent for a fixed annual income,—the income being payable in yearly, half-yearly, or quarterly instalments, according to agreement,—is said to be converted into an ANNUITY.

Although, however, usually employed in this restricted sense, the term "annuity" may be applied to any fixed annual income—such as a salary, a pension, &c.

303. Annuities are of two kinds—CERTAIN and CONTINGENT. An annuity is "certain" when it is to continue for a definite number of years (10, 20, 50, &c.—as the case may be), or for ever; but when the length of time depends upon the life of some particular person, or upon the life of the survivor of two or more persons, an annuity is a "contingent"—or a LIFE—annuity.

304. A "certain" annuity which is to continue for ever is called a PERPETUAL annuity, or a PERPETUITY.

305. When an annuity—either certain or contingent—becomes payable at once, it is said to be *immediate*, or *in possession*: when, on the other hand, it is not to be available until a certain period of time shall have elapsed, or until a future event (somebody's death, for instance) shall have occurred, an annuity is known as a *deferred*—or a *reversionary*—annuity.

Annuities "in possession" are sometimes spoken of as *annuities on lives;* and "reversionary" annuities, as *annuities on survivorship.*

306. When an annuity is allowed to remain unpaid for a certain length of time, the sum to which, for arrears and compound interest chargeable upon them, the annuitant becomes entitled is called the AMOUNT of the annuity.

307. By the PRESENT VALUE of an annuity is meant—the ready-money which, if improved at compound interest, would exactly pay the annuitant's claim in full.

If an annuity, falling due in annual instalments, remained unpaid for (say) 10 years, the annuitant would be entitled, not only to the arrears, but also to compound interest for

9 years	on the	1st instalment	—the instalment falling due at the *end* of the	1st year
8 „		2nd „		2nd „
7 „		3rd „		3rd „
6 „		4th „		4th „
5 „		5th „		5th „
4 „		6th „		6th „
3 „		7th „		7th „
2 „		8th „		8th „

and to a year's (simple) interest on the 9th instalment.*

If, therefore, the instalments were £1 each, and the rate of interest 3 per cent., the amount of the annuitant's claim for the 10 years would be (§ 297)—

$$\pounds 1 + \pounds 1{\cdot}03 + \pounds 1{\cdot}03^2 + \pounds 1{\cdot}03^3 + \pounds 1{\cdot}03^4 + \pounds 1{\cdot}03^5 + \pounds 1{\cdot}03^6 + \pounds 1{\cdot}03^7 + \pounds 1{\cdot}03^8 + \pounds 1{\cdot}03^9$$

£	
1	10th instalment + *no* interest
1·03	9th „ +1 year's interest
$1{\cdot}03^2$	8th „ +2 years' COMPOUND interest
$1{\cdot}03^3$	7th „ +3 „ „ „
$1{\cdot}03^4$	6th „ +4 „ „ „
$1{\cdot}03^5$	5th „ +5 „ „ „
$1{\cdot}03^6$	4th „ +6 „ „ „
$1{\cdot}03^7$	3rd „ +7 „ „ „
$1{\cdot}03^8$	2nd „ +8 „ „ „
$1{\cdot}03^9$	1st „ +9 „ „ „

* Upon the 10th instalment, due at the end of the 10th year, no *interest would be chargeable.*

It will be seen that the numbers 1, $1{\cdot}03$, $1{\cdot}03^2$, $1{\cdot}03^3$, &c. form a geometrical progression, which has $1{\cdot}03$ for common ratio; so that (§ 266) the sum of the terms is $\frac{1{\cdot}03^9 \times 1{\cdot}03 - 1}{1{\cdot}03 - 1} = \frac{1{\cdot}03^{10} - 1}{{\cdot}03}$.

If the instalments were half-yearly ones of £1 each, the amount would be—

$$\overset{£}{1} + \overset{£}{1{\cdot}015} + \overset{£}{1{\cdot}015^2} + \overset{£}{1{\cdot}015^3} \ldots\ldots\ldots\ldots\ldots + \overset{£}{1{\cdot}015^{19}}$$

£1	£1·015	£1·015²	£1·015³	£1·015¹⁹
20th instalment + *no* interest	19th ,, +1 half-year's interest	18th ,, +2 half-years' COMPOUND interest	17th ,, +3 ,, ,,	1st ,, +19 ,, ,,

The sum of the terms of this progression, which has $1{\cdot}015$ for common ratio, is $\frac{1{\cdot}015^{19} \times 1{\cdot}015 - 1}{1{\cdot}015 - 1} = \frac{1{\cdot}015^{20} - 1}{{\cdot}015}$.

If the instalments were quarterly ones of £1 each, the amount would be—

$$\text{£}1 + \text{£}1{\cdot}0075 + \text{£}1{\cdot}0075^2 + \text{£}1{\cdot}0075^3 \ldots\ldots\ldots\ldots + \text{£}1{\cdot}0075^{39}$$

40th instalment + *no* interest	39th ,, +1 quarter's interest	38th ,, +2 quarters' COMPOUND interest	37th ,, +3 ,, ,,	1st ,, +39 ,, ,,

The sum of the terms of this progression, which has $1{\cdot}0075$ for common ratio, is $\frac{1{\cdot}0075^{39} \times 1{\cdot}0075 - 1}{1{\cdot}0075 - 1} = \frac{1{\cdot}0075^{40} - 1}{{\cdot}0075}$.

308. To determine the amount of an unpaid annuity falling due in instalments of £1 each: Divide the interest of £1 for one "period"—*i.e.*, for a year, a half-year, or a quarter, as the case may be—into the compound interest of £1 for the given time.

Looking at the fractions $\frac{1{\cdot}03^{10} - 1}{{\cdot}03}$, $\frac{1{\cdot}015^{20} - 1}{{\cdot}015}$, and $\frac{1{\cdot}0075^{40} - 1}{{\cdot}0075}$, we see that—the rate being 3 per cent.—

$$\left.\begin{array}{l}\text{£}{\cdot}03\\ \text{£}{\cdot}015\\ \text{£}{\cdot}0075\end{array}\right\}\text{ is the interest of £1 for a }\left\{\begin{array}{l}\text{year}\\ \text{half-year}\\ \text{quarter}\end{array}\right.$$

Also, that the compound interest of £1 for 10 years is $1{\cdot}03^{10} - 1$, $1{\cdot}015^{20} - 1$, or $1{\cdot}0075^{40} - 1$, according as interest on principal is supposed to fall due in yearly, half-yearly, or quarterly *instalments.*

309. To determine the amount of an unpaid annuity falling due in instalments of P pounds each: Find what the amount would be if the instalments were £1 each (§ 308), and multiply this amount by P.

All other circumstances being the same, it is obvious that, whatever the amount may be for instalments of £1 each, the amount would be

$$\left.\begin{array}{l}\text{twice}\\ \text{3 times}\\ \text{4 ,,}\end{array}\right\}\text{ as much for instalments of }\left\{\begin{array}{l}\text{£2 each}\\ \text{£3 ,,}\\ \text{£4 ,,}\end{array}\right.$$
&c. &c.

EXAMPLE I.—An annuity, falling due in annual instalments of £36 15*s.* each, remained unpaid for 17 years; find the amount, at $3\frac{1}{2}$ per cent.

If the instalments were £1 each, the amount would be £$\frac{1{\cdot}035^{17}-1}{{\cdot}035}$. The required amount, therefore, is £$\frac{1{\cdot}035^{17}-1}{{\cdot}035}$ $\times 36{\cdot}75=$£$834{\cdot}408=$£834 8*s.* 2*d.*

EXAMPLE II.—An annuity, falling due in half-yearly instalments of £123 7*s.* 6*d.* each, remained unpaid for 13 years; find the amount, at 4 per cent.

If the instalments were £1 each, the amount would be £$\frac{1{\cdot}02^{26}-1}{{\cdot}02}$. The required amount, therefore, is £$\frac{1{\cdot}02^{26}-1}{{\cdot}02}$ $\times 123{\cdot}375=$£$4154{\cdot}16=$£4154 3*s.* $2\frac{1}{2}$*d.*

EXAMPLE III.—An annuity, falling due in quarterly instalments of £360 each, remained unpaid for 7 years; find the amount, at 5 per cent.

If the instalments were £1 each, the amount would be £$\frac{1{\cdot}0125^{28}-1}{{\cdot}0125}$. The required amount, therefore, is £$\frac{1{\cdot}0125^{28}-1}{{\cdot}0125}$ $\times 360=$£$11{,}980{\cdot}44=$£11,980 8*s.* $9\frac{1}{2}$*d.*

310. The "amount" of an annuity for a certain time, and at a certain rate, being known, to find the annuity itself: Divide the given amount by the amount of an annuity of £1 for the same time, and at the same rate.

This follows from § 309.

Example IV.—What annuity, falling due in annual instalments, would have amounted to £300 in 10 years, at 5 per cent.?

If the instalments were £1 each, the amount would be £$\frac{1\cdot05^{10}-1}{\cdot05}$=£12·578 (nearly). The required annuity, therefore, is £$\frac{300}{12\cdot578}$=£23·85=£23 17*s*.

Example V.—What sum must a person save every half-year in order that, the savings being invested at 3 per cent.—compound interest, he may be worth £2,000 in 16 years?

If the sum saved half-yearly were £1, the amount realized would be £$\frac{1\cdot015^{32}-1}{\cdot015}$=£40·688 (nearly); saving £1 every half-year, and investing the money at compound interest, being obviously the same as allowing an annuity which falls due in half-yearly instalments of £1 each to remain out at compound interest. The required sum, therefore, is £$\frac{2,000}{40\cdot688}$=£49·154 =£49 3*s*. 1*d*.

Example VI.—What sum must a person save quarterly in order that, the savings being invested at 8 per cent.—compound interest, he may be worth £1,000 in 6 years?

If the sum saved quarterly were £1, the amount realized would be £$\frac{1\cdot02^{24}-1}{\cdot02}$=£30·422. The required sum, therefore, is £$\frac{1,000}{30\cdot422}$=£32·87=£32 17*s*. 5*d*. (nearly.)

311. To determine the present value of £1 which is to become due at a future time: Find what £1, if invested as principal, would have amounted to, at compound interest, in the given time; and take the reciprocal of this amount.

At 4 per cent., for instance, £1, ready-money, is equivalent to £1·04 payable a year hence; so that the ready-money equivalent to £1 payable a year hence is the fourth term of the proportion—

(Payable in a year)		(Ready-money)		(Payable in a year)		(Ready-money)
£1·04	:	£1	::	£1	:	£$\frac{1}{1\cdot04}$.

Again: £1, ready-money, is equivalent to £1·04^2 payable in

2 years; so that the ready-money equivalent to £1 payable 2 years is the fourth term of the proportion—

(Payable in 2 years)		(Ready-money)		(Payable in 2 years)		(Ready-money)
$£1{\cdot}04^2$	:	£1	::	£1	:	$£\frac{1}{1{\cdot}04^2}$.

In like manner, £1, ready-money, is equivalent to £1·c payable in 3 years; so that the ready-money equivalent to ± payable in 3 years, is the fourth term of the proportion—

(Payable in 3 years)		(Ready-money)		(Payable in 3 years)		(Ready-money)
$£1{\cdot}04^3$	:	£1	::	£1	:	$£\frac{1}{1{\cdot}04^3}$.

It could be shown, in the same way, that the present val of £1 payable in

$$\left.\begin{array}{l} 4 \text{ years} \\ 5 \quad ,, \\ 6 \quad ,, \end{array}\right\} \text{is} \left\{\begin{array}{l} £\frac{1}{1{\cdot}04^4} \\ £\frac{1}{1{\cdot}04^5} \\ £\frac{1}{1{\cdot}04^6} \end{array}\right.$$

&c. &c.

If, however, interest on principal fell due half-yearly, t present value of £1 payable in

$$\left.\begin{array}{l} \text{a half-year} \\ 2 \text{ half-years} \\ 3 \quad ,, \\ 4 \quad ,, \end{array}\right\} \text{would be} \left\{\begin{array}{l} £\frac{1}{1{\cdot}02} \\ £\frac{1}{1{\cdot}02^2} \\ £\frac{1}{1{\cdot}02^3} \\ £\frac{1}{1{\cdot}02^4} \end{array}\right.$$

&c. &c.

And if interest on principal fell due quarterly, the prese value of £1 payable in

$$\left.\begin{array}{l} \text{a quarter} \\ 2 \text{ quarters} \\ 3 \quad ,, \\ 4 \quad ,, \end{array}\right\} \text{would be} \left\{\begin{array}{l} £\frac{1}{1{\cdot}01} \\ £\frac{1}{1{\cdot}01^2} \\ £\frac{1}{1{\cdot}01^3} \\ £\frac{1}{1{\cdot}01^4} \end{array}\right.$$

&c. &c.

312. To determine the present value of an annuity falling due in instalments of £1 each, and continuing for a certain number of "periods"—years, half-years, or quarters, as the case may be: Find the present value of £1 payable at the expiration of the given time (§ 311); subtract this present value from unity; and divide the remainder by the interest of £1 for one period.

Let us suppose that an annuitant, entitled to £1 a year for 12 years, sells his claim for ready-money when the rate of interest is 4 per cent. The purchaser of this claim is to receive £1 at the end of a year, another £1 at the end of 2 years, another £1 at the end of 3 years, and so on—the last £1 becoming due at the end of 12 years. The present value of the first pound is £$\frac{1}{1{\cdot}04}$; of the second pound, £$\frac{1}{1{\cdot}04^2}$; of the third pound, £$\frac{1}{1{\cdot}04^3}$; and so on—the present value of the last pound being £$\frac{1}{1{\cdot}04^{12}}$. So that the present value of the annuity is—

$$\frac{1}{1{\cdot}04}+\frac{1}{1{\cdot}04^2}+\frac{1}{1{\cdot}04^3}+\frac{1}{1{\cdot}04^4}+\frac{1}{1{\cdot}04^5}+\frac{1}{1{\cdot}04^6}+\frac{1}{1{\cdot}04^7}+\frac{1}{1{\cdot}04^8}+\frac{1}{1{\cdot}04^9}$$
$$+\frac{1}{1{\cdot}04^{10}}+\frac{1}{1{\cdot}04^{11}}+\frac{1}{1{\cdot}04^{12}}$$

These fractions form a geometrical progression, which has $\frac{1}{1{\cdot}04}$ for common ratio; and their sum is $\left\{\frac{1}{1{\cdot}04}-\frac{1}{1{\cdot}04^{12}}\times\frac{1}{1{\cdot}04}\right\}$ $\div\left\{1-\frac{1}{1{\cdot}04}\right\}$, or (both dividend and divisor being multiplied by $1{\cdot}04$) $\dfrac{1-\dfrac{1}{1{\cdot}04^{12}}}{{\cdot}04}$.

If the instalments of £1 each were half-yearly ones, the present value of the annuity would be—

$$\frac{1}{1{\cdot}02}+\frac{1}{1{\cdot}02^2}+\frac{1}{1{\cdot}02^3}\quad.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\quad+\frac{1}{1{\cdot}02^{24}}$$

or $\dfrac{1-\dfrac{1}{1{\cdot}02^{24}}}{{\cdot}02}$

And if the instalments of £1 each were quarterly ones, the present value of the annuity would be—

$$\frac{1}{1\cdot01}+\frac{1}{1\cdot01^2}+\frac{1}{1\cdot01^3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad +\frac{1}{1\cdot01^{48}}$$

$$\text{or} \frac{1-\frac{1}{1\cdot01^{48}}}{\cdot01}.$$

Here we observe that the present value of £1 payable in 12 years is $£\frac{1}{1\cdot04^{12}}$, or $£\frac{1}{1\cdot02^{24}}$, or $£\frac{1}{1\cdot01^{48}}$—according as interest on principal falls due yearly, half-yearly, or quarterly; also, that the interest of £1 for one period is

$$\left.\begin{array}{l}£\cdot04\\ £\cdot02\\ £\cdot01\end{array}\right\}\text{ when the period is a }\left\{\begin{array}{l}\text{year}\\ \text{half-year}\\ \text{quarter}\end{array}\right.$$

313. To determine the present value of an annuity falling due in instalments of P pounds each, and continuing for a certain time: Find what the present value would be if the instalments were £1 each; and multiply the result by P.

It is obvious that when—other circumstances being the same—the present value of an annuity of £1 is multiplied by

$$\left.\begin{array}{l}2\\ 3\\ 4\\ \text{\&c.}\end{array}\right\}\text{ we obtain the present value of an annuity of }\left\{\begin{array}{l}£2\\ £3\\ £4\\ \text{\&c.}\end{array}\right.$$

Example VII.—What is the present value of an annuity falling due in annual instalments of £24 each, and continuing for 7 years—money being worth 5 per cent.?

If the instalments were £1 each, the present value would be $£\frac{1-\frac{1}{1\cdot05^7}}{\cdot05}$. The required present value, therefore, is $£\frac{1-\frac{1}{1\cdot05^7}}{\cdot05}\times24=£138\cdot874$ (nearly) = £138 17*s*. 6*d*. (nearly.)

Example VIII.—What is the present value of an annuity falling due in half-yearly instalments of £72 each, and continuing for 13 years—money being worth 6 per cent.?

If the instalments were £1 each, the present value would be $£\frac{1-\frac{1}{1\cdot03^{26}}}{\cdot03}$. The required present value, therefore, is $£\frac{1-\frac{1}{1\cdot03^{26}}}{\cdot03}\times72=£1,287\cdot13=£1,287$ 2s. 7d.

EXAMPLE IX.—What is the present value of an annuity falling due in quarterly instalments of £120 17*s.* 6*d.* each, and continuing for 11 years—money being worth 3 per cent?

If the instalments were £1 each, the present value would be

$$£\frac{1-\frac{1}{1\cdot0075^{44}}}{\cdot0075}.$$ The required present value, therefore, is

$$£\frac{1-\frac{1}{1\cdot0075^{44}}}{\cdot0075}\times 120\cdot875 = £4{,}515\cdot89 = £4{,}515\ 17s.\ 9d.$$

314. The present value of a "perpetual" annuity of £1 is the reciprocal of the interest of £1 for one period.

At 4 per cent., for instance, the present value of a perpetual annuity falling due in annual instalments of £1 each is—

$$\frac{1}{1\cdot04}+\frac{1}{1\cdot04^2}+\frac{1}{1\cdot04^3}+\text{\&c., } \textit{ad infinitum.}$$

The sum of the terms of this progression (§ 266, Note 1) is

$$\frac{1}{1\cdot04}\div\left(1-\frac{1}{1\cdot04}\right)=\frac{1}{1\cdot04}\div\frac{\cdot04}{1\cdot04}=\frac{1}{1\cdot04}\times\frac{1\cdot04}{\cdot04}=\frac{1}{\cdot04}.$$

If the instalments of £1 each were half-yearly ones, the present value would be—

$$\frac{1}{1\cdot02}+\frac{1}{1\cdot02^2}+\frac{1}{1\cdot02^3}+\text{\&c., } \textit{ad infinitum,}$$

$$\text{or } \frac{1}{1\cdot02}\div\left(1-\frac{1}{1\cdot02}\right)=\frac{1}{1\cdot02}\div\frac{\cdot02}{1\cdot02}=\frac{1}{1\cdot02}\times\frac{1\cdot02}{\cdot02}=\frac{1}{\cdot02}.$$

And if the instalments of £1 each were quarterly ones, the present value would be—

$$\frac{1}{1\cdot01}+\frac{1}{1\cdot01^2}+\frac{1}{1\cdot01^3}+\text{\&c., } \textit{ad infinitum,}$$

$$\text{or } \frac{1}{1\cdot01}\div\left(1-\frac{1}{1\cdot01}\right)=\frac{1}{1\cdot01}\div\frac{\cdot01}{1\cdot01}=\frac{1}{1\cdot01}\times\frac{1\cdot01}{\cdot01}=\frac{1}{\cdot01}.$$

315. To determine the present value of a perpetual annuity falling due in instalments of P pounds each: Find what the present value would be if the instalments were £1 each, and multiply the result by P.

In other words: Divide P by the interest of £1 for one ***period.***

EXAMPLE X.—What is the present value of a perpetual annuity which falls due in annual instalments of £20 each—the rate of interest being $3\frac{1}{2}$ per cent.?

If the instalments were £1 each, the present value would be £$\frac{1}{\cdot035}$. The required present value, therefore, is £$\frac{1}{\cdot035}\times20$ $=$£$\frac{20}{\cdot035}=$£571 8*s.* 7*d.* (nearly.)

EXAMPLE XI.—What is the present value of a perpetual annuity which falls due in half-yearly instalments of £35 10*s.* each—the rate of interest being $4\frac{1}{4}$ per cent.?

If the instalments were £1 each, the present value would be £$\frac{1}{\cdot02125}$. The required present value, therefore, is £$\frac{1}{\cdot02125}$ $\times35\cdot5=$£$\frac{35\cdot5}{\cdot02125}=$£1,670 11*s.* 9*d.*

EXAMPLE XII.—What is the present value of a perpetuity which falls due in quarterly instalments of £144 12*s.* 6*d.* each—the rate of interest being $3\frac{3}{4}$ per cent.?

If the instalments were £1 each, the present value would be £$\frac{1}{\cdot009375}$. The required present value, therefore, is £$\frac{1}{\cdot009375}$ $\times144\cdot625=$£$\frac{144\cdot625}{\cdot009375}=$£15,426 13*s.* 4*d.*

316. To determine the present value of a "deferred" annuity: Find what the present value would be, if the annuity were immediate, (*a*) for the time which is to elapse before the annuity comes into possession, and (*b*) for the time which is to elapse before the annuity terminates; the difference between these two present values will be the present value required.

Thus, in determining the present value of a deferred annuity which is to come into possession 5 years hence, and then to continue for 25 years, we regard the annuity as immediate, and find its present values for 5 years and (5+25=) 30 years, *respectively*; the difference between these two present values is *evidently* the present value of the deferred annuity.

In practice, the answers to exercises like the preceding would be taken from an annuity table, which, at the present day, is always employed in the calculation of Contingent or Life annuities.

The principles involved in the construction of Life-annuity tables cannot be fully entered into here. It may be observed, however, that the *probable* duration of the life of any particular person is determined, by what is called the "Doctrine of Probabilities," from such materials as are supplied by the records of births and deaths in different localities; those materials being considered in connexion with the person's age, health, habits, occupation, &c. For instance, it has been ascertained that, of 10,000 infants born alive in Great Britain and Ireland, only—on an average—

6,460	live to the age of	10 years
6,090		20 ,,
5,642		30 ,,
5,075		40 ,,
4,397		50 ,,
3,643		60 ,,
2,401		70 ,,
953		80 ,,
142		90 ,,
9		100 ,,

Taking, therefore, the case of an ordinarily healthy person 30 years old, we find that his chances of living to the age of

$$\left.\begin{matrix}40\\50\\60\end{matrix}\right\} \text{ are } \left\{\begin{matrix}5{,}075\\4{,}397\\3{,}643\end{matrix}\right\} \text{ to } 5{,}642$$

&c. &c.

So that—ABSOLUTE CERTAINTY being represented by unity—the *probability* of his reaching the age of

$$\left.\begin{matrix}40\\[1ex]50\\[1ex]60\end{matrix}\right\} \text{ is measured by the fraction } \left\{\begin{matrix}\frac{5075}{5642}\\[1ex]\frac{4397}{5642}\\[1ex]\frac{3643}{5642}\end{matrix}\right.$$

&c. &c.

In order to understand the use which is made of such fractions as these, let us suppose that the person whose case we are considering purchases a life annuity—to be paid in annual *instalments* of £50 each. Now, if there were an absolute

certainty of his being alive at the end of 10 years, the present value of the 10th instalment—the rate of interest being, say, 3 per cent.—would be $£\frac{50}{1{\cdot}03^{10}}$; but as the payment of this instalment is contingent upon his living to claim it, and as the probability of his completing his 40th year is represented by $\frac{5075}{5642}$, it is evident that the present value of the 10th instalment is $\left(\text{not } £\frac{50}{1{\cdot}03^{10}}, \text{ but}\right) \frac{5075}{5642}$ of $£\frac{50}{1{\cdot}03^{10}}$. In like manner, the present value of the 20th instalment is $£\frac{50}{1{\cdot}03^{20}} \times \frac{4397}{5642}$; of the 30th instalment, $£\frac{50}{1{\cdot}03^{30}} \times \frac{3643}{5642}$; &c. The sum of the present values, found in this way, of all the instalments which the annuitant might *possibly* live to claim, constitutes the present value of the life annuity.

The present value of an annuity falling due in annual instalments of £1 each, and continuing during the annuitant's life, is technically spoken of as the VALUE of that life.

DIFFERENT SYSTEMS OF NOTATION.

It is by no means improbable that, instead of the DECIMAL system of notation, we should have a

$$\left.\begin{array}{l}\text{BINARY}\\ \text{TERNARY}\\ \text{QUATERNARY}\\ \text{QUINARY}\\ \&\text{c.}\end{array}\right\} \text{one if human beings were furnished with only} \left\{\begin{array}{l}\textit{two}\\ \textit{three}\\ \textit{four}\\ \textit{five}\\ \&\text{c.}\end{array}\right\} \text{fingers each.}$$

In order to realize any of these systems, we have merely to return to the illustration employed at page 3, and suppose that each of the boys there referred to has exactly the number of fingers indicated by the "base" of the system.

Thus, if the boys had only *two* fingers each, a finger held up by the

$$\left.\begin{array}{l}\textit{1st}\text{ boy}\\ \textit{2nd}\text{ ,,}\\ \textit{3rd}\text{ ,,}\\ \textit{4th}\text{ ,,}\\ \&\text{c.}\end{array}\right\} \text{would represent} \left\{\begin{array}{l}\textit{one}\text{ nut}\\ \textit{two}\text{ nuts}\\ \textit{four}\text{ ,,}\\ \textit{eight}\text{ ,,}\\ \&\text{c.}\end{array}\right.$$

So that, in a BINARY system, the values of the digit I would be—

	group of *eight*	group of *four*	group of *two*	UNIT	*half*	*fourth*	*eighth*	
. . .	I	I	I	I·	I	I	I	. . .

If the boys had only *three* fingers each, a finger held up by the

1st boy ⎫ ⎧ *one* nut
2nd ,, ⎬ would represent ⎨ *three* nuts
3rd ,, ⎪ ⎪ *nine* ,,
4th ,, ⎭ ⎩ *twenty-seven* ,,
&c. &c.

So that, in a TERNARY system, the values of I would be—

	group of *twenty-seven*	group of *nine*	group of *three*	UNIT	*third*	*ninth*	*twenty-seventh*	
. . .	I	I	I	I·	I	I	I	. . .

If the boys had only *four* fingers each, a finger held up by the

1st boy ⎫ ⎧ *one* nut
2nd ,, ⎬ would represent ⎨ *four* nuts
3rd ,, ⎪ ⎪ *sixteen* ,,
4th ,, ⎭ ⎩ *sixty-four* ,,
&c. &c.

So that, in a QUATERNARY system, the values of I would be—

	group of *sixty-four*	group of *sixteen*	group of *four*	UNIT	*fourth*	*sixteenth*	*sixty-fourth*	
. . .	I	I	I	I·	I	I	I	. . .

It is unnecessary to proceed further with these illustrations. We see that, in any system similar in principle to the decimal, 1 would have the following values—if the base were represented by B:

$$\ldots\; 1 = B^4 \quad 1 = B^3 \quad 1 = B^2 \quad 1 = B \quad 1\cdot = \text{UNITY} \quad 1 = \frac{1}{B} \quad 1 = \frac{1}{B^2} \quad 1 = \frac{1}{B^3} \quad 1 = \frac{1}{B^4} \;\ldots$$

And by doubling, or trebling, or quadrupling, &c. these values, we should, of course, obtain the corresponding values of 2, of 3, or of 4, &c.—as the case may be.

The number of different *digits* in any system would be less by 1 than the base of the system; the base indicating the number of different *figures*, and one of those figures being, in every instance, a *cipher*.* Thus, as the figures employed in the decimal system are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9—so, the figures employed in a

binary	system would be	0, 1
ternary		0, 1, 2
quaternary		0, 1, 2, 3
quinary		0, 1, 2, 3, 4
senary		0, 1, 2, 3, 4, 5
septenary		0, 1, 2, 3, 4, 5, 6
octary		0, 1, 2, 3, 4, 5, 6, 7
nonary		0, 1, 2, 3, 4, 5, 6, 7, 8

In a system having a higher base than 10, one or more new digits would be required. For instance: in an undenary system [base=eleven] there should be a character for *ten;* in a duodenary system [base=twelve] two new characters would be necessary—one for *ten,* and one for *eleven;* &c.

It is obvious that, in any system, the removal of the units' point (we can no longer say "decimal" point) n places, to the right or left, would multiply or divide—as the case may be—a combination of figures by B^n; the base of the system being represented by B.

---

* In the illustration at page 3, each boy is supposed, on finding all his fingers up, to "begin again"—being relieved (so to speak) by his neighbour, who puts up one finger. If, therefore, the boys were furnished with B fingers each, no boy would, at the close of the reckoning, have more than $B-1$ fingers up; so that, in a system of notation *having B* for base, the value of the highest digit would, in the units' *place, be* $B-1$.

The transposition of a number from the decimal to a different system, or *vice versa*, will be understood from the following examples :—

EXAMPLE I.—Transpose 1398 from the decimal to the quinary system.

Dividing 1398 as often as possible by 5 (the base of the quinary system), and annexing the several remainders—beginning with the last—to the last quotient, we find that, in the quinary system, the given number would be 21043; *i.e.*, 2 groups each $= 5^4$, 1 group $= 5^3$, [no group $= 5^2$], 4 groups each $= 5$, and 3 units :

$$\begin{array}{l} 5)\underline{1398} \\ \;5)\underline{279}+3 \text{ (units)} \\ \;\;5)\underline{55}+4 \text{ (groups each} = 5) \\ \;\;\;5)\underline{11}+0 \text{ (group} = 5^2) \\ \;\;\;\;\;2+1 \text{ (group} = 5^3) \end{array}$$

$1398=279\times5+3=(55\times5+4)\times5+3=55\times5^2+4\times5+3=(11\times5+0)\times5^2+4\times5+3=11\times5^3+0\times5^2+4\times5+3=(2\times5+1)\times5^3+0\times5^2+4\times5+3=2\times5^4+1\times5^3+0\times5^2+4\times5+3.$

EXAMPLE II.—Transpose ·864 from the decimal to the ternary system.

Here we have to convert a number—consisting of 8 tenths, 6 hundredths, and 4 thousandths—into *thirds*, *ninths*, *twenty-sevenths*, &c. This conversion is effected in pretty much the same way as the conversion of a decimal of a pound into shillings and pence. Multiplying ·864 by 3 (the base of the ternary system), we obtain 2 *thirds*, and a decimal (·592) of a third. Multiplying this decimal by 3, we obtain 1 *ninth*, and a decimal (·776) of a ninth. Multiplying the decimal of a ninth by 3, we obtain 2 *twenty-sevenths*, and a decimal (·328) of a twenty-seventh. Multiplying this last decimal by 3, we obtain 1 *eighty-first* (nearly). So that, in the ternary system, the given number would be ·2121.

$$\begin{array}{r} \cdot864 \\ \underline{3} \\ 2\cdot592 \\ \underline{3} \\ 1\cdot776 \\ \underline{3} \\ 2\cdot328 \\ \underline{3} \\ \cdot984 \end{array}$$

EXAMPLE III.—Transpose 23456·54321 from the decimal to the septenary system.

The integral portion of this number must be converted into groups of *seven*, groups of *forty-nine*, groups of *three hundred and forty-three*, &c. each ; and the decimal portion into *sevenths*, *forty-ninths*, *three hundred and forty-thirds*, &c. We therefore proceed as follows, and find that, in the septenary system, the given number would be 125246·354215 :

7)23456	·54321
7)3350+6	7
7)478+4	3·80247
7)68+2	7
7)9+5	5·61729
1+2	7
	4·32103
	7
	2·24721
	7
	1·73047
	7
	5·11329

EXAMPLE IV.—Transpose 21043 from the quinary to the decimal system.

Here we have—

$$\left.\begin{array}{rcr} 2\times 5^4 & = & 1250 \\ 1\times 5^3 & = & 125 \\ 4\times 5 & = & 20 \\ 3\times 1 & = & 3 \\ \hline \text{Ans.} & = & 1398 \end{array}\right\} \text{in the decimal system.}$$

This result can be obtained in a different way, as shown in the margin: multiplication by 5 (the base of the quinary system) converting groups each equal to

$$\left.\begin{array}{c} 5^4 \\ 5^3 \\ 5^2 \end{array}\right\} \text{into groups each equal to} \left\{\begin{array}{c} 5^3 \\ 5^2 \\ 5 \end{array}\right.$$

and converting groups of 5 each into units.

$$\begin{array}{r} 21043 \\ 5 \\ \hline 11 \\ 5 \\ \hline 55 \\ 5 \\ \hline 279 \\ 5 \\ \hline 1398 \end{array}$$

EXAMPLE V.—Transpose ·2121 from the ternary to the decimal system.

Multiplying ·2121 by 10, we obtain 8 *tenths*, and a portion (·1221) of a tenth. Multiplying ·1221 by 10, we obtain 6 *hundredths*, and a portion (·1021) of a hundredth. Lastly, multiplying ·1021 by 10, we obtain 4 *thousandths*. So that, in the decimal system, the given number would be ·864.

$$\begin{array}{r} ·2121 \\ 10 \\ \hline 8·1221 \\ 10 \\ \hline 6·1021 \\ 10 \\ \hline 4·0121 \end{array}$$

NOTE.—In performing these multiplications, we "carry" 1 *for every three*, instead of for every ten—the base of the ternary *system* being 3.

EXAMPLE VI.—Transpose 125246·354215 from the septenary to the decimal system.

The last two examples suggest the way in which this transposition is to be effected. In dealing with the integral portion of the number, however, we employ 7 as multiplier, instead of 5; and, in dealing with the non-integral portion, we "carry" 1 for every 7, instead of for every 3—the base of the septenary system being 7. We thus find that, in the decimal system, the given number would be 23456·54321 (nearly):

125246	·354215
7	10
9	5·301131
7	10
68	4·215033
7	10
478	3·131462
7	10
3350	2·042666
7	10
23456	0·621654

The following exercises in the fundamental operations—addition, subtraction, multiplication, and division—are worked out for the information of the student. In both the undenary and the duodenary system, *t* is employed to represent *ten*; and, in the latter system, *e* is put for *eleven* :—

ADDITION.

Septenary system.	*Quinary* system.	*Octary* system.
56342	142·3	60·157
12635	314·2	51·762
34513	234·1	13·036
23450	420·3	21·643
10261	341·2	40·254
204164	3113·1	227·316

SUBTRACTION.

Senary system.	*Quaternary* system.	*Nonary* system.
402351	3102·13	653·813
340512	1230·32	274·635
21435	1211·21	368·167

MULTIPLICATION.

Ternary system.	*Senary* system.	*Undenary* system.
210121	13045	435·78*t*
2	23	9
1121012	43223	3597·432
	30134	
	345003	

DIVISION.

Quinary system.	*Septenary* system.	*Duodenary* system.
3)42023	5)436125	8)9547·*t6e*
12141	62432·254	1220·*e9t*

If we consider the advantages and disadvantages of different systems of notation, we shall find that, to be desirable, a system should have neither a very large nor a very small base, and should, moreover, be as free as possible from "circulators"—*i.e.*, from non-integral numbers corresponding to what, in the decimal system, are known as circulating decimals.

If the base were very large, arithmetical operations would involve too much *mental* labour. In a vigesimal system, for instance [base=twenty], there would be a character for every (integral) number below twenty, and the multiplication table would have to be carried as far as "twenty times"; whilst, in a sexagesimal system [base=sixty], there would be a character for every number below sixty, and the multiplication table would have to be continued to "sixty times."

On the other hand, if the base were very small, numbers would not be expressed with sufficient conciseness: they would occupy an inconveniently large number of places; so that a perplexing multiplicity of names would be necessary, and calculations, besides being very tedious, would involve too much *mechanical* labour. Thus, the comparatively small number *three hundred and sixty-five* would, in the binary system, be written 101101101; in the ternary system, 111112; in the quaternary system, 11231; &c.

With regard to "circulators," it will be seen that as, in the decimal system, the only fractions convertible into terminate decimals are those having, or reducible to others having, powers *of ten for* denominators—so, in any other system, the only *fractions* whose values could be expressed by non-repeating figures

to the right of the units' place, would be those having, or reducible to others having, powers of the *base* (whatever it may be) for denominators. It will also be seen that if any number, not a measure of the base, occurred amongst the prime factors of the denominator of a fraction which had been reduced to its simplest form, the fraction would not be reducible to one having a power of the base for denominator; just as, in the decimal system, a fraction, to be reducible to a terminate decimal (or to a decimal fraction), must, when in its simplest form, have in its denominator no prime factor different from 2 and 5.

In either the senary or the duodenary system, a fraction, in order not to give rise to a circulator, should, when in its lowest terms, have in its denominator no prime factor different from 2 and 3—in other words, should have for denominator a power of 2, a power of 3, or the product of a power of 2 by a power of 3; and as such denominators would obviously form a larger class, and occur more frequently, than the corresponding denominators in the decimal system, it follows that there would be fewer circulators in either the senary or the duodenary than there are in the decimal system, which, however, gives rise to a much smaller number of circulators than would occur in any other system except the two just mentioned.

In the binary, the quaternary, or the octary system, a fraction would give rise to a circulator if the denominator—the fraction being in its simplest form—contained any other prime factor than 2; that is, if the denominator were not a power of 2. Again, in the ternary or the nonary system, a fraction would give rise to a circulator if the denominator—the fraction being in its simplest form—contained any other prime factor than 3; that is, if the denominator were not a power of 3. In like manner, the only "non-circulator" denominators (as they may be called) in the quinary system would be powers of 5; in the septenary system, powers of 7; in the undenary system, powers of 11; &c. It thus appears that the only systems to which the decimal is not decidedly superior are the senary and the duodenary. Of these two, the senary would, on the whole, answer quite as well as the decimal; and the duodenary would answer better. In the senary system, the highest digit would be 5, and the multiplication table would be carried only as far as "six times:" so that, although numbers would be expressed less concisely, calculations would involve less mental labour, in the senary than in the decimal system; and besides, as we have seen, the senary system would give rise to fewer circulators than occur in the decimal system.

In the duodenary system, which would be as free from circulators as the senary, numbers would be expressed more concisely, whilst the mental labour involved in calculations would

be very little greater, than in the decimal system. Indeed, as it is, we always carry the multiplication table as far as "twelve times."

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THE END.

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